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q-deformation and semidualization in 3D quantum gravity

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Abstract

We explore in detail the role in euclidean 3D quantum gravity of quantum Born reciprocity or ‘semidualization’. The latter is an algebraic operation defined using quantum group methods that interchanges position and momentum. Using this we are able to clarify the structural relationships between the effective noncommutative geometries that have been discussed in the context of 3D gravity. We show that the spin model based on $D(U(su_2))$ for quantum gravity without cosmological constant is the semidual of a quantum particle on a 3-sphere, while the bicrossproduct (DSR) model based on $\mathbb{C}[\mathbb{R}^2 \rtimes \mathbb{R}] \rtimes U(su_2)$ is the semidual of a quantum particle on hyperbolic space. We show further how the different models are all specific limits of q -deformed models with $q = e^{-\hbar\sqrt{-\Lambda}/m_p}$, where m_p is the Planck mass and Λ is the cosmological constant, and argue that semidualization interchanges $m_p \leftrightarrow l_c$, where l_c is the cosmological length scale $l_c = 1/\sqrt{|\Lambda|}$. We investigate the physics of semidualization by studying representation theory. In both the spin model and its semidual we show that irreducible representations have a physical picture as solutions of a respectively noncommutative/curved wave equation. We explain, moreover, that the q -deformed model, at a certain algebraic level, is self-dual under semidualization.

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1. Introduction

Whatever quantum gravity actually is, it must provide classical continuum geometry at macroscopic scales and involve corrections at the Planck scale. In recent years, it has become more widely accepted that these corrections should, at least at first order, be described by

some kind of noncommutative geometry in which coordinate algebras are noncommutative or ‘quantum’. A useful setting for exploring this idea is provided by 3D quantum gravity, which is not a fully dynamical theory as in four dimensions but is a theory where many computations can be done in detail. In particular, one should be able to see in this theory exactly how noncommutative spacetime could emerge as a next-to-classical correction to conventional commutative spacetime. At the moment there are several candidate models for such noncommutative spacetimes even in the 3D setting. Our goal in this paper is to bring all of these models into a single coherent picture, to explain precisely the relationships between the models at the structural level and to explore their physical implications to some extent. One important lesson we learn is that these relationships emerge only in the full theory *with* cosmological constant, as different degenerations related by a Hopf algebraic duality operation of ‘semidualization’. Since we are mainly interested in the algebraic relationships, we focus on the Euclidean signature for simplicity, deferring the Lorentzian case to a sequel.

Of the various models, the most studied is the ‘spin model’, which is just the algebra of angular momentum but viewed as a noncommutative spacetime coordinate algebra. Its emergence as an effective spacetime for 3D quantum gravity without cosmological constant was anticipated in [1, 2]. It was put forward in [3] in view of its quantum symmetry group $D(U(su_2))$, whose role in 3D quantum gravity was proposed in [4] and established in [5]. The explicit emergence of this noncommutative spacetime starting from the Ponzano–Regge action was recently demonstrated in [6]. The q -deformation of this model, which, for q a root of unity, is the state sum behind the Turaev–Viro model, describes 3D quantum gravity *with* cosmological constant as controlled by the quantum group $D(U_q(su_2))$. The q -deformed local spacetime here is the quantum group $U_q(su_2)$ viewed as a noncommutative coordinate algebra.

Other models of spacetime noncommutativity have been proposed, which do not have a firmly established relation to quantum gravity. In this paper, we are particularly interested in the ‘bicrossproduct models’ introduced in the euclidean form in [7] and in 3+1 form in [8], related to the construction of what was called κ -Poincaré symmetry in [9]. The 3+1 bicrossproduct model is sometimes called ‘deformed special relativity’ but this is misleading as there are several other deformations of special relativity under consideration, and we therefore keep the more specific name. This model is of particular interest because it predicts an energy-dependent speed of light which will be tested by time of flight data currently being collected at the NASA Fermi gamma-ray space telescope (formerly GLAST). Note, however, that there is little evidence of a theoretical link between the bicrossproduct model and quantum gravity. In particular, it was recently shown [10] that the (2+1)-dimensional version of the bicrossproduct model (with a timelike noncommutative direction) does not arise directly in 3D quantum gravity. One of the upshots of the current paper is that bicrossproduct models do have a precise role related to quantum gravity in its usual presentation, via our semidualization map, or in physical terms by an interchange of position and momentum.

Also in the 1990s there was completely developed a q -deformed Minkowski space theory in the form of 2×2 braided Hermitian matrices [11]. We will show that these various models are all intimately related. To do this we use new results as well as results known to experts in quantum groups, and explained, for example, in [12]. A subsidiary purpose of this paper is to advertise some of those results to the quantum gravity community, where they are not so well known (with notable exceptions, see e.g. [13]).

In order to give an overview of our findings we need to look at the physical constants that enter quantum gravity, namely the gravitational constant G , Planck’s constant \hbar and the cosmological constant Λ (we work in units where the speed of light is 1). In 3D gravity, the

dimension of G is that of an inverse mass; the Planck mass is entirely classical and given by

$$m_p = \frac{1}{G}. \tag{1}$$

The cosmological constant has the dimension of inverse length squared and can be used to define a cosmological length scale l_c via

$$l_c = \frac{1}{\sqrt{|\Lambda|}}. \tag{2}$$

A second length scale is given by the Planck length, which takes the form

$$l_p = \hbar G = \frac{\hbar}{m_p}. \tag{3}$$

The dimensionless parameter q which plays the role of the deformation parameter in this paper is related to the ratio of the two length scales l_p and l_c . More precisely it is given by

$$q = e^{-\hbar G \sqrt{-\Lambda}}. \tag{4}$$

Note that this expression is specific to the euclidean theory we are considering in this paper; in the Lorentzian version one should replace Λ by its negative in the above expression, as explained in [14].

In order to organize the various models and symmetries appearing in this paper, we begin with the case where all three physical constants \hbar , G and Λ are non-zero. The quantum group $D(U_q(su_2))$, with q defined as in (4), plays an important role in euclidean 3D quantum gravity with a non-vanishing cosmological constant [15]. One can take the limit $q \rightarrow 1$ in several ways, with different physical interpretations. The first is to take $\hbar \rightarrow 0$, keeping G and Λ fixed. This gives an obviously classical gravity theory with cosmological constant, so that $\hbar = 0$ but $l_c < \infty$ and $m_p < \infty$. We will not be interested in this first limit and will in fact set $\hbar = 1$. A second way of taking the limit is to let $G \rightarrow 0$, keeping \hbar and Λ fixed. This gives a theory without gravitational self-interactions but with a cosmological constant, so that $l_c < \infty$ and $m_p = \infty$; the symmetry quantum group of this model is $U(so_{1,3})$ and gravity is effectively a classical background on which a quantum particle propagates. A third possibility is to take $\Lambda \rightarrow 0$ while keeping \hbar and G fixed, leading to a quantum gravity theory without cosmological constant i.e. $m_p < \infty$ and $l_c = \infty$; the symmetry quantum group is now $D(U(su_2))$. The joint limit $G \rightarrow 0$ and $\Lambda \rightarrow 0$ with $\hbar \neq 0$ is a free quantum particle propagating in euclidean space, controlled by the group E_3 of euclidean motions.

None of these limits give the bicrossproduct models. Instead we need the semidualization operation mentioned earlier. This comes out of quantum group theory and was used to understand both the quantum double and bicrossproducts. In general, semidualization takes any quantum group built from factors (in our case momentum and rotations) acting on some other space (in our case position space) and swaps the roles of position and momentum. We will elaborate this in detail later, but for now we only need to know that an original quantum group $H_1 \bowtie H_2$ acting on H_2^* semidualizes to a bicrossproduct quantum group $H_2^* \blacktriangleright H_1$ acting on H_2 , assuming there is an appropriate notion of dual cf [12, 16]. It is important to note that not only do position and momentum get swapped, the quantum group also gets changed so this is a change of model and not merely a (quantum) Fourier transform of the same model.

The quantum groups arising as limits of $D(U_q(su_2))$ and their semiduals are listed in table 1 for $\Lambda \leq 0$, together with the physical regimes to which they are associated. The table also shows that the values of the physical constants associated with semidual models are related by the exchange

$$m_p \leftrightarrow l_c. \tag{5}$$

Table 1. The quantum groups arising in 3D gravity for $\Lambda \leq 0$ and their semiduals. The diagonal entries are self-dual, up to a quantum Wick rotation in the $q \neq 1$ case.

| 3D gravity | | |
|----------------|---|--|
| | $m_p = \infty$ | $m_p < \infty$ |
| $l_c = \infty$ | $U(e_3) = U(su_2) \ltimes \mathbb{C}[\mathbb{R}^3]$ | $D(U(su_2)) = U(su_2) \ltimes \mathbb{C}[SU_2]$ |
| $l_c < \infty$ | $U(so_{1,3}) = U(su_2) \ltimes U(su_2^*)$ | $D(U_q(su_2)) \cong U_q(so_{1,3})$ |
| Semidual model | | |
| | $m_p = \infty$ | $m_p < \infty$ |
| $l_c = \infty$ | $U(e_3) = U(su_2) \ltimes \mathbb{C}[\mathbb{R}^3]$ | $U(su_2) \otimes U(su_2) = U(so_4)$ |
| $l_c < \infty$ | $U(su_2) \ltimes \mathbb{C}[SU_2^*]$ | $U_q(su_2) \otimes U_{q^{-1}}(su_2) = U_q(so_4)$ |

Interestingly, this duality does not involve \hbar , since both m_p and Λ are purely classical. Moreover, still assuming $\Lambda \leq 0$, we note that we can write the deformation parameter q in (4) as

$$q = e^{-\frac{\hbar}{m_p l_c}}. \tag{6}$$

This is invariant under the duality (5). Thus, according to the table quantum gravity with cosmological constant covered by $D(U_q(su_2))$ is in a certain algebraic sense self-dual: it is invariant under semidualization up to q -Wick rotation. This near self-duality is lost when one takes the limits $l_c \rightarrow \infty$ and $m_p \rightarrow \infty$ separately, but reappears when both limits are taken together: quantum theory of a free particle in euclidean space without cosmological constant, controlled by E_3 , is structurally invariant under semidualization and self-dual in this sense. Note that the requirement of self-duality or Born reciprocity requires that m_p and l_c are either both infinite (the E_3 flat space model) or both finite (the q -deformed model). Hence self-duality as an approach to quantum gravity, as advocated in [17], forces the cosmological constant to be non-zero.

Armed with this overview we can now outline the paper. Section 2 contains background material on Poisson Lie groups, a summary of the Chern–Simons formulation of 3D gravity and an explanation of the concept of semidualization for Hopf algebras.

Section 3 contains a detailed explanation and elaboration of the structural relations between the Hopf algebras summarized in table 1. We describe each of the Hopf algebras in detail, and give precise definitions of the various limits, semidualization maps and isomorphisms that relate them. The general statement, made earlier in this introduction, that semidualization swaps the role of positions and momenta is elaborated in this section and illustrated by examples. An important role in this section is played by isomorphisms like the one between $D(U_q(su_2))$ and $U_q(so_{1,3})$ (indicated by \cong in table 1) which are ‘purely quantum phenomena’ in the sense that they only hold when $q \neq 1$. Taking the limit $q \rightarrow 1$ on either side of such an isomorphism gives different quantum groups, and this provides the mathematical definition of the physical distinction between taking the limit $l_c \rightarrow \infty$ and the limit $m_p \rightarrow \infty$. A key finding of this section is the result, already sketched above, that 3D quantum gravity with cosmological constant is self-dual up to q -Wick rotation. We also explain why this near self-duality *fails* in the limit $q \rightarrow 1$. The reason is that a ‘purely quantum’ isomorphism used in the near self-duality breaks down when $q = 1$ and therefore, when one takes the limit $q \rightarrow 1$, one can do it on either side of the isomorphism and will have different theories.

In section 4, we explore the physical meaning of semidualization in greater detail. Ultimately we would like to understand this operation and the ‘self-duality’ under it in the full $q \neq 1$ theory, but the latter is at present too poorly understood at this level of detail. However, with the aid of noncommutative geometry we do obtain a clear picture in the degenerate cases. Our starting point for the physical interpretation in all cases is the fact that fundamental

symmetries of physics enter quantum theoretical models via their representations. Thus the Klein–Gordon, Dirac and Maxwell equations all determine irreps of the Poincaré group, and the free Schrödinger equation determines an irrep of the Galilei group. This applies in our models on the local ‘model spacetime’ on which our quantum symmetry groups act, which is a part of the information in the theory (it has to be supplemented by patching information according to the topology). Our strategy is therefore to study representations of a model and its semidual, and to compare them.

Next, we have said that semiduality interchanges position and momentum. So on one hand we have particles moving on position space and forming a representation of our quantum symmetry group, and in the semidual model we have waves on what in the original model was called momentum space. We can use Fourier transform to map over the physics of the semidual model over to our original position space in order to compare with the original model, and we do this. Thus our original position space has two kinds of fields on it. One is a set of particles forming irreducible representations of the original quantum symmetry group and the other is a second set of fields forming an irreducible representation of the semidual quantum symmetry group. Note that not only are position and momentum swapped under semidualization but the quantum symmetry group also changes as we have seen in table 1. Secondly, when the position space is classical but curved its Fourier dual is a noncommutative space, and vice versa, i.e. we need methods of quantum Fourier transform [12] and noncommutative differential geometry in order to establish this picture.

It is instructive here to start with the trivial case of the group E_3 , which we do in section 4.1. The semidual theory is also controlled by E_3 but with position and momentum interchanged. The structure is self-dual in this sense, with duality implemented by the \mathbb{R}^3 Fourier transform, but of course the actual physics of interest is not. Physical states are elements of irreps of E_3 , but are realized quite differently on the two sides of the semiduality. As expected, an irrep of E_3 on one side consists of waves in position space, obeying a first-order differential constraint and the wave equation. But on the other side it consists of monopole sections on spheres of increasing radius in position space. The two ‘physical models’ here are equivalent under Fourier transform *and* an exchange of position and momentum. We express the monopole sections in terms of a linear vector-valued function obeying an algebraic constraint and show that the algebraic constraint maps to the differential constraint under Fourier transform. This itself is quite interesting and is explained in detail.

In sections 4.2 and 4.3, we look at the similar semiduality between the $D(U(su_2))$ spin model (3D quantum gravity without cosmological constant) and a quantum particle on SU_2 with the action of $SU_2 \times SU_2$ from the left and the right. We start in section 4.2 with the $D(U(su_2))$ model and $U(su_2)$ as the noncommutative or ‘fuzzy’ position space. The group SU_2 then plays the role of a curved momentum space. We show how to describe irreps of $D(U(su_2))$ in terms of vector-valued functions on this (curved) momentum space, obeying an algebraic constraint. A quantum group Fourier transform [3, 18, 19] maps these to solutions of noncommutative wave equations. For spins 0, 1/2 and 1 we recover the known [3] noncommutative wave equations on the spin-model noncommutative (‘fuzzy’) \mathbb{R}^3 . Our approach can in principle be extended to obtain fuzzy wave equations of all spin.

Then, in section 4.3, we turn to the semidual model and write the irreps of $SU_2 \times SU_2$ in terms of vector-valued functions on SU_2 (now interpreted as curved position space) which obey a differential equation. This time, a noncommutative Fourier transform gives us a picture of the irreps for this model as noncommutative monopole sections on fuzzy spheres in noncommutative momentum space. The physics in this model is not the same as the physics in the previous model of which it is the semidual. For example, the physical momentum values

labelling the irreps are now discrete whereas before they were continuous. However, they have a ‘similar form’ as a remnant of the near self-duality in the full q -deformed theory.

This exemplifies the general construction. The semidual model, by construction, has its representations on a space which is the (quantum Fourier or Hopf algebra) dual of the space where the original model has its representations (in the discussion above, the original model is represented on H_2^* with Fourier dual H_2 , which is the space where the semidual model is represented). So one always has one space where fields of both models live, which is functions on position space for one model and functions on momentum space for the other. In order to compare the two models further, we fix the interpretation of this space, as fields on position space, say. Then irreps of one (quantum) group are realized by means of a wave equation constraint and irreps of the semidual (quantum) group by means of an algebraic (projective module) constraint. In the case of sections 4.2 and 4.3, the space for one model is the angular momentum algebra and its dual is that of functions on SU_2 . However, unlike in the E_3 case, the (quantum) groups which are being represented in the two cases are different. Indeed, the models are different: one is quantum gravity without cosmological constant and the other is a quantum particle with cosmological constant. In the q -deformed case we return to the quantum groups being algebraically (twisting) equivalent although still with different unitarity $*$ -structure requirements. These remarks are developed further in section 5. The appendix contains a summary of facts about forms and vector fields on Lie groups in our conventions.

Remark on units. Most of this paper is concerned with quantum-mechanical methods applied on classical backgrounds or in quantum gravity. As a rule we therefore set $\hbar = 1$. To revert to physical units the reader should insert \hbar every time a mass is expressed in terms of an inverse length or a length in terms of an inverse mass.

2. Background: 3D gravity and quantum groups

Here we provide the background in both physics and mathematics that we need for our analysis. After a short summary of Poisson Lie group theory we review classical 3D gravity, using the language of Poisson Lie groups. We explain the role of quantum doubles in 3D quantum gravity and review key features. Finally, we introduce the semidualization functor and study some of its properties.

2.1. Poisson Lie groups

We write \mathfrak{g} for the Lie algebra of a Lie group G . When we require explicit generators we use a basis in which the structure constants are purely imaginary. In the case of G being unitary, this means that the generators are Hermitian, with real eigenvalues, simplifying our discussions of representation theory and quantum mechanics. Additional results and conventions regarding the differential geometry of Lie groups, which are needed later in this paper, are summarized in the appendix.

A Poisson Lie group means a Lie group G which is a Poisson manifold, so there is a Poisson bracket among smooth functions on G , such that the product map $G \times G \rightarrow G$ is a map of Poisson spaces. Here $G \times G$ has the direct product Poisson manifold structure. It is known that such a Poisson bracket is equivalent to a map $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ at the Lie algebra level, called the Lie cobracket. It is just the adjoint of the Poisson bracket $\mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ when restricted to $\mathfrak{g}^* \subset C^\infty(G)$. The pair (\mathfrak{g}, δ) with appropriate axioms is called a Lie bialgebra and should be thought of as an infinitesimal quantum group. A Poisson Lie group is quasitriangular if $\delta\xi = \text{ad}_\xi(r)$ where $r \in \mathfrak{g} \otimes \mathfrak{g}$ obeys the classical Yang–Baxter equation and

its symmetric part r_+ is ad-invariant. It is called *factorizable* if it is quasitriangular and r_+ is non-degenerate as a map $\mathfrak{g} \rightarrow \mathfrak{g}^*$. The associated Poisson Lie group is similarly factorizable in this situation (either locally near the identity or, with appropriate technical assumptions, globally). For any Lie bialgebra there is a double $d(\mathfrak{g}) = \mathfrak{g} \bowtie \mathfrak{g}^{*\text{op}}$ which is factorizable as is its Poisson Lie group $d(G) = G \bowtie G^{*\text{op}}$ where $G^{*\text{op}}$ is the opposite (with reversed product) of the Lie group associated with the dual Lie bialgebra \mathfrak{g}^* . We will use \star to denote the combination $^{*\text{op}}$. This group and G are both subgroups and the formula $su = (s \triangleright u)(s \triangleleft u)$ defines the ‘dressing action’ \triangleright of G on $G^* = G^{*\text{op}}$. The action \triangleleft the other way is called the ‘backreaction’ or dual dressing action. These matters and the general \bowtie theory which they relate to were explained in [20], where one of us proved a theorem that Lie splitting data exponentiate whenever one factor is compact. This theorem holds for general factorizations not limited to the double or ‘Manin triple’.

Note that since $d(G)$ is factorizable, its dual $d(G)^*$ is a Poisson Lie group that is diffeomorphic to $d(G)$, at least near the identity, via a map

$$Z : d(G)^* = G^* \blacktriangleright G \rightarrow d(G)$$

given in this case canonically by multiplication in $d(G)$. Under this map orbits in $d(G)^*$ under the dressing action of $d(G)$ map over to conjugacy classes in $d(G)$ as spaces. We will use the symplectic structure on these orbits, which are symplectic leaves for the Poisson bracket on $d(G)^*$.

Quantum groups such as $\mathbb{C}_q[G]$ are quantizations of G with its standard Drinfeld–Sklyanin Poisson bracket, defined for all semisimple Lie groups. Their duals $U_q(\mathfrak{g})$ deform the classical enveloping algebras $U(\mathfrak{g})$ and can also, with a bit of care, be viewed as quantization of the Drinfeld dual G^* [21]. The quantization of $d(G)^*$ can be viewed as yielding $D(U_q(\mathfrak{g}))$ i.e. the quantum double construction for quantum groups to be described in detail later.

2.2. Reminder of 3D gravity with point sources

We consider gravity in three dimensions coupled to matter in the form of a fixed number of point particles and review the Chern–Simons formulation of the theory. For simplicity, we restrict attention to three-dimensional manifolds of the form $\Sigma \times \mathbb{R}$, where Σ is a closed two-dimensional manifold of genus γ and with n marked points, one for each point particle. Concentrating on the euclidean version, we view gravity in a first-order form of a dreibein e^a , where $a = 1, 2, 3$, and a spin connection ω with values in so_3 . These data can be combined together into a single \mathfrak{g} -valued gauge field A , where \mathfrak{g} is one of the following: the Lie algebra e_3 of the euclidean group E_3 (for vanishing cosmological constant), the Lie algebra $sl_2(\mathbb{C}) \cong so_{3,1}$ of $SL_2(\mathbb{C})$ (for negative cosmological constant) and the Lie algebra so_4 of $SU(2) \times SU(2)$ (for positive cosmological constant). In the following, we write \mathcal{G} for any of the three associated simply connected Lie groups and Λ for the cosmological constant. Introducing generators P_a of translations and generators J_a of rotations, with commutation relations

$$[J_a, J_b] = \iota \epsilon_{abc} J_c, \quad [P_a, J_b] = \iota \epsilon_{abc} P_c, \quad [P_a, P_b] = \iota \Lambda \epsilon_{abc} J_c, \tag{7}$$

the spin connection can be expanded $\omega = -\iota \omega_a J^a$ and the gauge field A is

$$A = -\iota(e_a P_a + \omega_a J_a).$$

In order to define an action principle for this connection one requires a non-degenerate, invariant symmetric bilinear form k on the Lie algebra \mathfrak{g} . In terms of the generators above this is given by

$$k(J_a, P_b) = -\frac{m_p}{8\pi} \delta_{ab}, \tag{8}$$

with all other pairings of generators giving zero. The standard Chern–Simons action for the connection A , formulated with the symmetric form k , then reproduces the Einstein–Hilbert action in the first-order formalism, as observed by Achucarro and Townsend [22] and elaborated by Witten [23]. The constant $m_p/(8\pi)$, which is related to Newton’s constant via (1), is not normally included in the symmetric form k but instead kept as a coupling constant which multiplies the Chern–Simons action. However, since the non-degenerate symmetric bilinear form ultimately determines the Poisson structure on the phase space of the theory, the inclusion of the physical constants here makes it easier to keep track of them in subsequent calculations.

The physical degrees of freedom of Chern–Simons theory are encoded in the \mathcal{G} -valued holonomies of the connection A as follows. To each puncture i we associate an element $\xi_i^* \in \mathfrak{g}^*$ encoding the mass m_i and spin s_i of the particle i via

$$\xi_i^* = \iota(m_i P_3^* + s_i J_3^*)$$

in a dual basis. Using the form (8) we obtain an associated element in \mathfrak{g} :

$$\xi_i = -\iota \frac{8\pi}{m_p} (m_i J_3 + s_i P_3). \tag{9}$$

The curvature of the connection A has a delta-function singularity at each puncture i with coefficients lying in the adjoint orbit of ξ_i . Correspondingly, the holonomy around the puncture i is forced to lie in the conjugacy class C_i containing e^{ξ_i} . The extended phase space is

$$\tilde{P} = \mathcal{G}^{2\gamma} \times \prod C_i \tag{10}$$

and the actual phase space is

$$P = \{(A_\gamma, B_\gamma, \dots, A_1, B_1, M_i) \in \tilde{P} \mid [A_\gamma, B_\gamma^{-1}] \cdots [A_1, B_1]^{-1} \prod M_i = 1\} / \text{Ad}(\mathcal{G}).$$

A_i, B_i are holonomies around and through handles, while M_i are holonomies around our punctures, all with reference to some arbitrary base point $*$. The reader may wonder here where in the moduli space is the location of our n marked points at any given time. The answer is that the physics is diffeomorphism invariant so to a large extent these are irrelevant. Correspondingly, all that we retain from Σ in P is its topology. However, one can say a bit more about ‘positions’ of the particles in the theory. To do this we need to consider the Poisson structure of the theory.

The gauge groups \mathcal{G} of the Chern–Simons formulation of gravity are all Poisson Lie groups. The Poisson structure does not enter into the formulation of the gauge theory, but plays an important role in describing the Poisson structure of its phase space, as we shall explain. We focus on two here, both arising in the euclidean situation (later on we will suggest two more). Without cosmological constant, we take

$$\mathcal{G} = d(SU_2) = SU_2 \ltimes su_2^* = E_3 \tag{11}$$

as a group but with a non-trivial Poisson bracket. Here SU_2 here is regarded as a Poisson Lie group with the zero Poisson bracket and we then take its double. Hence su_2^* is a Lie algebra with zero Lie bracket and hence we can also view it as an Abelian group, with the Kirillov–Kostant Poisson bracket. With negative cosmological constant, we take

$$\mathcal{G} = d(SU_2) = SU_2 \bowtie SU_2^* = SL_2(\mathbb{C}) \tag{12}$$

as a group but with a non-trivial Poisson structure. Here SU_2 is a Poisson Lie group equipped with its Drinfeld–Sklyanin bracket and we take its double.

There is a natural Poisson structure on \tilde{P} given by a certain ‘braided tensor product’ of those on each copy of $\mathcal{G} \times \mathcal{G}$ and on each conjugacy class [24] which descends to the Atiyah–Bott one on P . In the Hamiltonian approach (see [25–27], and [15] in the context of 3D gravity), its quantization is the main step in constructing quantum gravity coupled to point sources. Equivalently the braidings can be untangled and \tilde{P} is Poisson equivalent to the direct product of the Poisson structures on the conjugacy classes C_i and the Heisenberg–double ones on γ copies of $\mathcal{G} \times \mathcal{G}$ [28]. We concentrate on the former, associated with the punctures. The conjugacy classes C_i in \mathcal{G} are the image under a bijection

$$Z : \mathcal{G}^* \rightarrow \mathcal{G},$$

discussed in section 2.1, of the symplectic leaves of the Poisson structure on \mathcal{G}^* . The map is provided by an invariant, non-degenerate symmetric bilinear form at the level of the associated Lie bialgebras (assuming again that we work with the associated connected and simply connected Lie groups, or ignore certain global issues).

To proceed further, we make use of the fact that the Poisson Lie groups discussed so far are all (special cases of) double crossproducts $\mathcal{G} = G_1 \bowtie G_2$ of Poisson Lie groups (this means that they factorize into the two Poisson Lie subgroups and can be recovered from them by means of a double semidirect product in which each G_1 and G_2 acts on the set of the other and with the direct product Poisson structure). Then $\mathcal{G}^* = G_1^* \blacktriangleleft G_2^*$ (a direct product as groups and a certain double-semidirect Poisson structure). One can describe the inverse images $Z^{-1}(C_i)$ in these terms. If the Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 of G_1 and G_2 have generators J_a, P_a respectively (not necessarily the same as in (7)), the dual Poisson Lie group has Lie algebra generators J_a^*, P_a^* , say, forming a dual basis to these (so that $\langle J_a^*, J_b \rangle = \langle P_a^*, P_b \rangle = \delta_{ab}$). The coefficients in these bases form a local coordinate system for \mathcal{G}^* near the identity which we shall use, namely $j_a = \langle -\iota J_a, (\cdot) \rangle$ is $-\iota J_a$ as linear functions on \mathfrak{g}_1^* and $p_a = \langle -\iota P_a, (\cdot) \rangle$ as linear functions on \mathfrak{g}_2^* . One may then write the Poisson bracket of \mathcal{G}^* explicitly among j_a and p_a . When restricted to $Z^{-1}(C_i)$ they form the classical phase space coordinates associated with each conjugacy class.

Also, $\mathcal{G} = G_1 \bowtie G_2$ acts canonically on the dual Poisson Lie group G_2^* (say) and one can form a crossproduct ‘Heisenberg–Weyl group’ $(G_1 \bowtie G_2) \bowtie G_2^*$. In physics this group should be represented in the quantum algebra of observables, i.e. its enveloping algebra as a quantization of the dual Poisson manifold $(G_1^* \blacktriangleleft G_2^*) \blacktriangleright G_2$ as an extended phase space. Here this copy of G_2 has coordinates near the identity which we denote now by $x_a = \langle \iota P_a^*, (\cdot) \rangle$ as linear functions on \mathfrak{g}_2 . One has then additional Poisson brackets for these variables among themselves and with the previous j_a, p_a . We shall prove these facts at the Hopf algebra level in section 3 and the Poisson Lie versions follow analogously.

To see all of this explicitly and also to understand the physical role of these ‘position variables’ x_a , we concentrate on the case of vanishing cosmological constant, so $\mathcal{G} = E_3 = SU(2) \bowtie su_2^*$. Our conventions for this group are spelled out in section 3.1; note that they differ from those used in a similar context in [5, 29]. The group \mathcal{G}^* is simply the direct product $E_3^* = su_2^* \times SU_2$ according to what we have said above. The map Z is

$$Z(\vec{j}, u) = (u, \text{Ad}_u^*(\vec{j})),$$

where we use our above bases for \mathfrak{g}_1^* and \mathfrak{g}_2 in each case: $j = \iota \vec{j} \cdot \vec{J}^*$ is an element of $\mathfrak{g}_1^* = su_2^*$ on the left and $-\frac{8\pi}{m_p} \iota \text{Ad}_u^*(\vec{j}) \cdot \vec{P}$ is an element of $\mathfrak{g}_2 = su_2^*$ on the right³. Meanwhile, P_a^* obey the rescaled su_2 commutation relations

$$[P_a^*, P_b^*] = -\iota \frac{8\pi}{m_p} \epsilon_{abc} P_c^*.$$

³ For the Abelian Lie group su_2^* , the Lie algebra coordinates provide global coordinates on the group.

In view of the non-degenerate symmetric bilinear form (8) on e_3 we could identify

$$P_a^* \leftrightarrow -\frac{8\pi}{m_p} J_a, \quad (13)$$

but will refrain from doing so to avoid confusion. Thus $u = e^{iP_a P_a^*}$ in terms of our local coordinates for G_2^* near the identity.

Let us focus on one conjugacy class C containing the element e^ξ with ξ parametrized as in (9) (and the index i dropped). As we shall explain below, one can describe the preimage $Z^{-1}(C)$ of a conjugacy class C in \mathcal{G} as the subset of elements $(j, u) \in \mathcal{G}^*$ with coordinates obeying the further constraints

$$\vec{p}^2 = m^2, \quad \vec{j} \cdot \vec{p} = ms. \quad (14)$$

The Poisson structure of \mathcal{G}^* gives rise to the brackets

$$\{j_a, j_b\} = \epsilon_{abc} j_c, \quad \{j_a, p_b\} = \epsilon_{abc} p_c, \quad \{p_a, p_b\} = 0, \quad (15)$$

and it is easy to check that the combinations (14) are Casimirs, confirming that the conjugacy classes are indeed the symplectic leaves of the Poisson structure (15). The Poisson brackets suggest that we should think of p_a as the particle's momentum and j_a as the particle's 'angular momentum' coordinates. However, the coordinates p_a fail for the group element u when $|\vec{p}| = m_p/4$ and $u = -1$. Thus, in 3D gravity we should really interpret u as the particle's group-valued momentum. Momentum space is curved and has the structure of a non-Abelian Lie group. This is a classical effect and means that, even classically, momentum addition is noncommutative.

Geometrically, the space of vectors \vec{p} and \vec{j} obeying the constraints (14) parametrizes the space of all lines in \mathbb{R}^3 , and we shall see next that we may think of these lines as the particle's worldline in an auxiliary euclidean space with the coordinates x_a . Thus, if we describe a symplectic leaf of \mathcal{G}^* over in $\mathcal{G} = E_3$ as a conjugacy class, we can redundantly parametrize it in terms of elements $(g, x) \in E_3$ that occur in $C = \{(g, x)^{-1} e^\xi (g, x)\}$. The image under Z of the point (\vec{j}, u) in the physical phase space obeying (14) maps over redundantly to a set of points $(g, x) \in E_3$ such that $Z(\vec{j}, u) = (g, x)^{-1} e^\xi (g, x)$. This set of points is described by $g \in G_1 = SU_2$ and a coordinate vector \vec{x} for $x = -i\vec{x} \cdot \vec{P} \in G_2 = su_2^*$ obeying

$$\vec{j} = \frac{m_p}{8\pi} (\text{Ad}_{u^{-1}}^* - 1)(\vec{x}) + s \frac{\vec{p}}{m}, \quad \text{Ad}_{g^{-1}}(m J_3) = \vec{p} \cdot \vec{J}.$$

Note that we have identified the translation part of the group E_3 with the position in the auxiliary euclidean space by fixing an origin. The limit

$$(\text{Ad}^*(u^{-1}) - 1)\vec{x} \approx \frac{8\pi}{m_p} \vec{x} \times \vec{p}$$

for small m/m_p suggests, by analogy with the flat-space formula for angular momentum, that we should interpret \vec{x} as the particle's (spacetime) position. Further support for this interpretation comes from the following geometrical consideration. Position coordinates should act on momentum space by translation. Since, as we just saw, momentum space is curved, such translations cannot commute if they are to be globally defined. One finds that

$$\{x_a, x_b\} = -\frac{8\pi}{m_p} \epsilon_{abc} x_c, \quad (16)$$

as well as

$$\{j_a, x_b\} = \epsilon_{abc} x_c, \quad \{x_a, f\} = -\frac{8\pi}{m_p} \xi_a^R(f) \quad (17)$$

for the Poisson brackets with the coordinates of \mathcal{G}^* . Here f is any function on $G_2^* = SU_2$ and ξ_a^R is the right-translation vector field associated with the generator J_a of the Lie algebra according to (A.2). The geometrical meaning of these brackets is that the Poisson brackets of position coordinates are those of the su_2 Lie algebra, and that they act on the momentum manifold SU_2 as generators of right multiplication. Note that the bracket (16) is also part of the initial Poisson structure on $\mathcal{G} = E_3$ (with all other brackets vanishing in our case). The conjugation action of E_3 on conjugacy classes is the dressing action on symplectic leaves of \mathcal{G}^* ; this is a Poisson action with the Poisson structure of \mathcal{G} taken into account.

The above discussion reveals Poisson noncommutativity of position coordinates in 3D gravity, but there are important caveats. First of all, we can change the coordinate vector \vec{x} to $\vec{x} + \tau \frac{\vec{p}}{m}$, where τ is an arbitrary real parameter, without changing the vectors \vec{p} and \vec{j} . This is in agreement with our interpretation of \vec{p} and \vec{j} as parameters of a worldline: shifting the position vector along the worldline does not change the worldline itself. The second, and more important, caveat is that all of the above coordinates refer to the extended phase space \tilde{P} and are therefore not well defined on the physical phase space P . One may interpret them as referring to an auxiliary Euclidean space associated with the base point $*$ where the holonomies start and end. However, to obtain the physical phase space we should divide by Euclidean motions in that space. The Poisson brackets of physical quantities like traces of (products of) holonomies have been studied in [30], but the relation with the above position coordinates has not been clearly established. An alternative approach is to study universes with boundary. In that case there is a preferred family of ‘centre-of-mass frame’ of the universe. By choosing the base point to be associated with one such frame, the coordinates of the holonomies with respect to the base point regain some of their physical meaning. This approach is pursued in [29, 31, 32].

The above description of the phase of 3D gravity in terms of the Poisson Lie structures associated with \mathcal{G} is tailormade for the Hamiltonian approach to the quantization of Chern–Simons theory [25–27]. In this approach, a key role is played by a Hopf algebra H which is a quantization of the Drinfeld dual \mathcal{G}^* . The Hilbert space of the quantized Chern–Simons theory can then be described in terms of representation theory of H in a manner which is analogous to the construction of the classical phase space as a quotient of the extended phase space (10). Schematically (and referring to the above references for details) the quantization of the extended phase space is a tensor product of γ copies of a representation \mathcal{R} of H , which is the analogue of the regular representation of a group (and the quantization of the Heisenberg double Poisson manifold $\mathcal{G} \times \mathcal{G}$), and irreps V_i of H for each of the punctures (the quantization of the conjugacy classes C_i). The Hilbert space of the quantized Chern–Simons theory is

$$\mathcal{H} = \text{Inv}(\mathcal{R}^{\otimes \gamma} \otimes \bigotimes V_i), \tag{18}$$

where Inv denotes the H -invariant part of the tensor product. For the cases of Euclidean gravity without (11) and with negative cosmological constant (12), the relevant quantum groups are the quantum doubles $D(U(su_2))$ and $D(U_q(su_2))$ ($q \in \mathbb{R}$). Details of the Hamiltonian quantization programme for these cases can be found, respectively, in [15, 33].

2.3. Quantum double and semidualization theorem

Having motivated the role here of quantum groups in the picture, we now fix our notations for these and recall the quantum double. Let H be a Hopf algebra over \mathbb{C} , with coproduct $\Delta : H \rightarrow H \otimes H$, counit $\epsilon : H \rightarrow \mathbb{C}$ and antipode $S : H \rightarrow H$. The particular real form of interest is expressed by, in addition, a $*$: $H \rightarrow H$ making H into a Hopf $*$ -algebra. We let H' be a suitable dual of H such that it is also a Hopf algebra and dually paired with H by

a non-degenerate map $\langle h, a \rangle$. We refer to [12] for all further details. It is useful to use the ‘Sweedler notation’ $\Delta h = h_{(1)} \otimes h_{(2)}$.

The quantum double $D(H) = H \bowtie H'^{\text{op}}$ is built on the vector space $H \otimes H'$ with new product

$$(h \otimes a) \cdot (g \otimes b) = hg_{(2)} \otimes ba_{(2)} \langle g_{(1)}, a_{(1)} \rangle \langle Sg_{(3)}, a_{(3)} \rangle,$$

where $h, g \in H, a, b \in H'$, and the tensor product coproduct [16, 34]. This Hopf algebra has a canonical action [12] on H

$$h \triangleright g = h_{(1)}gSh_{(2)}, \quad a \triangleright h = \langle a, h_{(1)} \rangle h_{(2)}$$

and induces on it the canonical braid statistics

$$\Psi(h \otimes g) = h_{(1)}gSh_{(2)} \otimes h_{(3)}$$

with respect to which H is Ψ -commutative. It also induces braid statistics on any other objects covariant under $D(H)$. There is a canonical action of $D(H)$ on H which we can therefore view as a ‘noncommutative space’ (assuming the Hopf algebra H is noncommutative). The dual of the quantum double is $H' \blacktriangleright H^{\text{cop}}$ which means the tensor product as an algebra (its coproduct is twisted). It contains the ‘noncommutative position algebra’ H which ties in with our semiclassical picture above.

If H is cocommutative i.e. H' commutative we have $D(H) = H \ltimes H'$ with

$$(h \otimes a)(g \otimes b) = hg_{(1)} \otimes a \triangleleft g_{(2)} \cdot b, \quad a \triangleleft g = a_{(2)} \langle g, a_{(1)} \rangle Sa_{(3)}$$

i.e. the semidirect product by the right coadjoint action corresponding to the left adjoint coaction of H' on itself, see [12] for the Hopf algebra formalism.

2.4. Semidualization

The general construction of which the quantum double is part is a ‘double crossproduct’ $H = H_1 \bowtie H_2$ of a Hopf algebra factorizing into two sub-Hopf algebras. Factorizing means that the map $H_1 \otimes H_2 \rightarrow H$, given by viewing in H and multiplying there, is an isomorphism of linear spaces. In this situation one deduces actions $\triangleright : H_2 \otimes H_1 \rightarrow H_1$ and $\triangleleft : H_2 \otimes H_1 \rightarrow H_2$ of each Hopf algebra on the vector space of the other. These are defined by $(1 \otimes a) \cdot (h \otimes 1) = (a_{(1)} \triangleright h_{(1)} \otimes a_{(2)} \triangleleft h_{(2)})$ for the product of H viewed on $H_1 \otimes H_2$. The coproduct of $H_1 \bowtie H_2$ is the tensor one given by the coproduct on each factor and there is a canonical action of this Hopf algebra on the vector space of H_1 by

$$(h \otimes a) \triangleright f = h \cdot (a \triangleright f), \quad \forall f \in H_1, \quad h \otimes a \in H_1 \otimes H_2.$$

This in fact respects the coalgebra structure of H_1 and hence provides in a canonical way a covariant right action of $H_1 \bowtie H_2$ on H'_1 as an algebra. Explicitly, the right action of H_2 on H'_1 is defined by

$$\langle \phi \triangleleft a, h \rangle = \langle \phi, a \triangleright h \rangle, \quad \forall \phi \in H'_1, \quad a \in H_2, \quad h \in H_1,$$

and in these terms the right action of $H_1 \bowtie H_2$ on H'_1 is

$$\phi \triangleleft (h \otimes a) = \langle \phi_{(1)}, h \rangle \phi_{(2)} \triangleleft a.$$

In this case, we may form the crossproduct algebra by this action

$$(H_1 \bowtie H_2) \ltimes H'_1. \tag{19}$$

Also in this situation we may dualize one of the factors, say replacing H_1 by H'_1 . This gives a new Hopf algebra $H_2 \blacktriangleright H'_1$ (the semidual of H) which then acts covariantly from the left on H_1 as an algebra. The product and coproduct are

$$(a \otimes \phi)(b \otimes \psi) = ab_{(1)} \otimes \phi \triangleleft b_{(2)} \psi, \quad \Delta(a \otimes \phi) = (a_{(1)} \otimes a_{(2)(0)} \phi_{(1)}) \otimes (a_{(2)(1)} \otimes \phi_{(2)})$$

$$a \triangleleft h = \langle a_{(0)}, h \rangle a_{(1)}, \quad h \in H_1, \quad a, b \in H_2, \quad \phi, \psi \in H'_1,$$

where the coaction on $a \in H_2$ is defined in terms of our original \triangleleft . Its canonical left action on H_1 is

$$(a \otimes \phi) \triangleright h = a \triangleright h_{(1)} \langle \phi, h_{(2)} \rangle.$$

This is the ‘semidualization functor’ that applies to Hopf algebras that factorize [12, 16]. In this case we have a crossproduct algebra by the action on H_1 :

$$H_1 \rtimes (H_2 \bowtie H'_1). \tag{20}$$

Lemma 2.1. *The two algebras (19) and (20) are the same when built in the vector space $H_1 \otimes H_2 \otimes H'_1$. Hence there is one algebra*

$$A = (H_1 \bowtie H_2) \bowtie H'_1 = H_1 \rtimes (H_2 \bowtie H'_1)$$

independently of the point of view, with

$$H_1 \bowtie H_2 \subset A \supset H_2 \bowtie H'_1$$

i.e., containing both the double crossproduct and the bicrossproduct. Moreover, $A \supset H_1 \bowtie H'_1 = H_1 \rtimes H'_1$ the Heisenberg–Weyl algebra.

Proof. This is automatic from the definition of the semidualization process when one goes into how this is actually defined by dualizing the involved actions and coactions. Indeed, the product of A computed the first way is

$$\begin{aligned} (h \otimes a \otimes \phi) \cdot (g \otimes b \otimes \psi) &= (h \otimes a) \cdot (g \otimes b)_{(1)} \otimes (\phi \triangleleft (g \otimes b)_{(2)}) \cdot \psi \\ &= (h \otimes a) \cdot (g_{(1)} \otimes b_{(1)}) \otimes \langle \phi_{(1)}, g_{(2)} \rangle (\phi_{(2)} \triangleleft b_{(2)}) \cdot \psi \\ &= h \cdot (a_{(1)} \triangleright g_{(1)}) \otimes (a_{(2)} \triangleleft g_{(2)}) \cdot b_{(1)} \otimes \langle \phi_{(1)}, g_{(3)} \rangle (\phi_{(2)} \triangleleft b_{(2)}) \cdot \psi. \end{aligned}$$

Meanwhile, computing the product the other way gives

$$\begin{aligned} (h \otimes a \otimes \phi) \cdot (g \otimes b \otimes \psi) &= h \cdot ((a \otimes \phi)_{(1)} \triangleright g) \otimes (a \otimes \phi)_{(2)} \cdot (b \otimes \psi) \\ &= h \cdot ((a_{(1)} \otimes a_{(2)(0)} \phi_{(1)}) \triangleright g) \otimes (a_{(2)(1)} \otimes \phi_{(2)}) \cdot (b \otimes \psi) \\ &= h \cdot (a_{(1)} \triangleright g_{(1)}) \langle a_{(2)(0)} \phi_{(1)}, g_{(2)} \rangle \otimes a_{(2)(1)} b_{(1)} \otimes (\phi_{(2)} \triangleleft b_{(2)}) \cdot \psi \\ &= h \cdot (a_{(1)} \triangleright g_{(1)}) \langle a_{(2)(0)}, g_{(2)} \rangle \langle \phi_{(1)}, g_{(3)} \rangle \\ &\quad \otimes a_{(2)(1)} b_{(1)} \otimes (\phi_{(2)} \triangleleft b_{(2)}) \cdot \psi, \end{aligned}$$

which is the same on using the definition of the coaction on H_2 . Also, the product restricted to $h \otimes \phi = h \otimes 1 \otimes \phi$ is

$$(h \otimes \phi) \cdot (g \otimes \psi) = hg_{(1)} \otimes \langle \phi_{(1)}, g_{(2)} \rangle \phi_{(2)} \psi,$$

which can be viewed either way $H_1 \bowtie H'_1 = H_1 \rtimes H'_1$ as a crossproduct of the coregular representation (in the finite-dimensional case it is the matrix algebra $\text{End}(H_1)$ [12]). \square

This gives a concrete rotation–momentum–position algebra way of thinking about semidualization. The three form a single algebra. If we think of H_1, H_2 as momentum, rotations we see the double crossproduct acting on H'_1 as positions, and if we think of H_2, H'_1 as rotations, momentum, we see the bicrossproduct acting on H_1 as positions. This is a version of ‘quantum Born reciprocity’ (interchanging position and momentum) which is a little different from the original motivation for bicrossproducts as quantum phase spaces, but based on entirely the same Hopf algebra dualization constructions namely to interchange H_1

with H'_1 . If one looks only at the position–momentum sector then this is the usual Heisenberg–Weyl algebra (sometimes called the ‘Heisenberg double’) which is symmetric between position and momentum so already admits the quantum Born reciprocity.

Finally, we can do the exact same constructions with the roles of H_1, H_2 swapped. Thus, there is similarly a canonical right action of $H_1 \bowtie H_2$ on the coalgebra of H_2 and its dualization is a canonical left action on the algebra of H'_2 . We can form a crossproduct by this. Alternatively, we can use the left action of H_1 on H'_2 and a right coaction of H'_2 on H_1 corresponding to \triangleright to define a bicrossproduct $H'_2 \blacktriangleright H_1$ which acts from the right on the algebra of H_2 . As before, we have

$$B = H'_2 \bowtie (H_1 \bowtie H_2) = (H'_2 \blacktriangleright H_1) \bowtie H_2$$

as an algebra

$$H_1 \bowtie H_2 \subset B \supset H'_2 \blacktriangleright H_1$$

within which the semidualization takes place. It contains $H'_2 \bowtie H_2 = H'_2 \blacktriangleleft H_2$. We will actually use the A -version of semidualization, given in lemma 2.1, in order that the bicrossproducts act naturally from the left, but this means that the double crossproduct acts naturally from the right. In the primary 3D quantum gravity models we prefer the B -version so that the double acts naturally from the left, but then the bicrossproduct acts from the right. To study their semiduals we flip conventions and use the A -model so that it is the bicrossproducts which act from the left (this is because physicists tend to avoid right actions in actual computations).

In particular, if one applies the second version of the semidualization (with dualizing algebra B) to $D(H) = H \bowtie H'^{\text{op}}$ one has the canonical Schrödinger left action on $(H'^{\text{op}})' = H^{\text{cop}} = H$ as an algebra as mentioned above. According to the above, we also have

$$B = H \bowtie D(H) = M(H) \bowtie H'^{\text{op}} \cong (H^{\text{cop}} \otimes H) \bowtie H'^{\text{op}}$$

for some ‘mirror product’ bicrossproduct

$$M(H) = H^{\text{cop}} \blacktriangleright H \cong H^{\text{cop}} \otimes H,$$

which as stated turns out to be isomorphic to the tensor product Hopf algebra [16]. In effect, the quantum Mach principle or semidualization (used the other way) converts something trivial over to something non-trivial, namely the quantum double, and was our way to construct it. The action of $H^{\text{cop}} \otimes H$ on H'^{op} from the right is

$$a \triangleleft (h \otimes g) = \langle h, a_{(1)} \rangle a_{(2)} \langle Sg, a_{(3)} \rangle$$

when one traces through the explicit constructions and isomorphisms. Note that $H \subseteq D(H)$ appears in $H^{\text{cop}} \otimes H$ embedded on the diagonal via the coproduct. Its right action is therefore evaluation against the left adjoint coaction of H' on itself. Likewise, if we use the A -version in order to have a left action here, and start with $D(H) = H'^{\text{op}} \blacktriangleleft H$ acting from the right on H^{cop} as an algebra, then the semidual is $H \blacktriangleleft H^{\text{cop}} \cong H \otimes H^{\text{cop}}$ acting on H'^{op} from the left by

$$(h \otimes g) \triangleright a = \langle Sh, a_{(1)} \rangle a_{(2)} \langle g, a_{(3)} \rangle.$$

All operations in these formulae refer to the underlying Hopf algebra H or its dual.

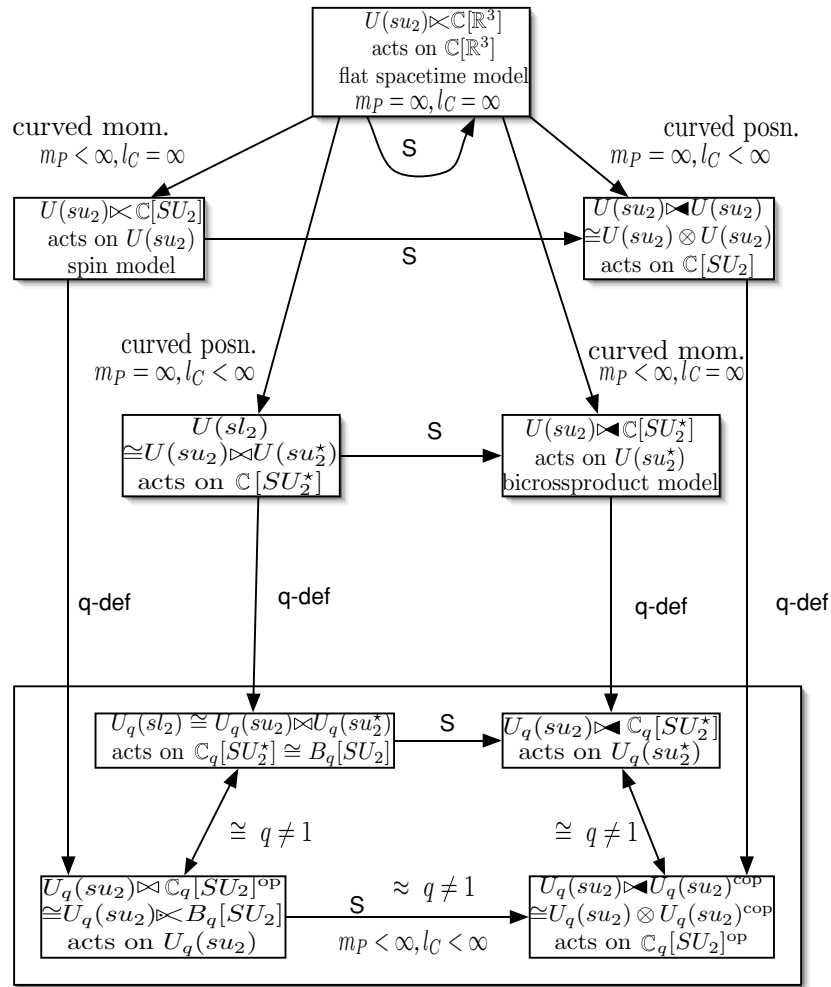


Figure 1. Overview of isometry quantum groups in euclidean 3D quantum gravity models (left) and their semiduals (right). Here SU_2 is a 3-sphere, SU_2^* is hyperbolic space, $U(su_2)$ and $U(su_2^*)$ are noncommutative versions of \mathbb{R}^3 . We denote semidualization by S .

3. Structure of the models as limits of 3D quantum gravity

After the above background, we describe in detail potentially eight noncommutative spacetime models for the eight entries in table 1. At this stage we are interested in the structure of the symmetry algebras of the models and at this level describe isomorphisms which reduce our models to only six. The more detailed situation is shown in figure 1, as we shall explain in this section.

We will also introduce explicit notations for our examples. We clarify first an important piece of notation. In physics, the word momentum can be used in two ways: (a) with reference to a point in momentum space $\vec{p} \in \tilde{\mathbb{R}}^3$ or (b) as an observable, which means its components P_a are particular functions on momentum space. When Lie symmetries are realized they usually appear in the second form. For example $U(\mathbb{R}^3)$, with generators P_a acting on the

algebra $\mathbb{C}[\mathbb{R}^3]$ of functions on position space by $P_a = -\iota \frac{\partial}{\partial x_a}$, is also the polynomial algebra $\mathbb{C}[P_1, P_2, P_3] = \mathbb{C}[\tilde{\mathbb{R}}^3]$ of functions on momentum space. In this point of view, P_1 is an infinitesimal element of \mathbb{R}^3 in the direction $(1, 0, 0)$ etc; it is a tangent vector in the Lie algebra of \mathbb{R}^3 and not a function on it. Rather, each P_a is a function on $\tilde{\mathbb{R}}^3$. This is clearer perhaps in the non-Abelian case where $U(\mathfrak{g})$ acts naturally by vector fields on $C^\infty(G)$, so elements of \mathfrak{g} here are tangent not cotangent vectors. At the same time, they are functions on cotangent space. Finally, although we will not often make this distinction, one can think of $\vec{p} = (p_1, p_2, p_3)$ not as an actual numerical point but as a generic point, i.e. as a placeholder for actual but unspecified points in momentum space $\tilde{\mathbb{R}}^3$. As soon as one does this, p_a becomes a coordinate function on $\tilde{\mathbb{R}}^3$, i.e. acquires the same status as P_a . Thus, it will often be useful (and would be normal in physics) to mix notations in this way in order to avoid a proliferation of symbols.

3.1. $E_3 = SU_2 \ltimes \mathbb{R}^3$ —free particle without cosmological constant (flat spacetime)

We actually work with the double cover of the euclidean group of motions in three dimensions:

$$E_3 = SU_2 \ltimes \mathbb{R}^3, \quad (21)$$

where we view SU_2 with the zero Poisson bracket and \mathbb{R}^3 denotes the translation group with zero Lie bracket and zero Poisson bracket. The vanishing of the Lie bracket (commutativity of spacetime translations) amounts to taking the cosmological constant to be zero (or, by (2), $l_c = \infty$) and the zero Poisson bracket on E_3 corresponds to a vanishing gravitational coupling constant (or, by (1), $m_p = \infty$). The action of SU_2 is by rotations which can be expressed concisely as

$$(g, a)(h, b) = (gh, \text{Ad}_h^*(a) + b), \quad g, h \in SU_2, \quad a, b \in \mathfrak{su}_2^*, \quad (22)$$

where we identify our Abelian translation group as $\mathbb{R}^3 = \mathfrak{su}_2^*$. We denote as before the generators of \mathfrak{su}_2^* by P_a . We assume these generators to be proportional to the duals J_a^* of the \mathfrak{su}_2 generators J_a , but not necessarily equal to them. The reason for this is that different normalizations of J_a relative to P_a are required in different contexts, see e.g. (8) in relation to 3D gravity. The upshot is that P_a form an orthogonal basis of \mathfrak{su}_2^* and that an element $a \in \mathfrak{su}_2^*$ can be written in terms of a coordinate vector \vec{a} as $a = -\iota \vec{a} \cdot \vec{P}$ in our conventions. The coadjoint action here is a right action defined by $\text{Ad}_h^*(a) : k \mapsto a(h(k)h^{-1})$, for $k \in \mathfrak{su}_2$, which we can also write by abuse of notation as $\text{Ad}_{h^{-1}}(\vec{a})$. In terms of the coordinate vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$ for the \mathfrak{su}_2^* -elements a and b the above multiplication law is thus

$$(g, \vec{a})(h, \vec{b}) = (gh, \text{Ad}_{h^{-1}}(\vec{a}) + \vec{b}). \quad (23)$$

By definition we also view the generators P_a as coordinates on momentum space, generating its commutative coordinate algebra. The momentum space itself is the Lie algebra \mathfrak{su}_2 as another copy of \mathbb{R}^3 .

The Lie algebra $e_3 = \mathfrak{su}_2 \ltimes \mathbb{R}^3 = \mathfrak{su}_2 \ltimes \mathfrak{su}_2^*$ has rotation generators J_a and translation generators P_a with commutation relations

$$[J_a, J_b] = \iota \epsilon_{abc} J_c, \quad [P_a, J_b] = \iota \epsilon_{abc} P_c, \quad [P_a, P_b] = 0. \quad (24)$$

Note that this Lie algebra is *not* a classical double since \mathfrak{su}_2^* here has the zero Lie cobracket, and its enveloping algebra $U(e_3) = U(\mathfrak{su}_2) \ltimes U(\mathfrak{su}_2^*) = U(\mathfrak{su}_2) \ltimes \mathbb{C}[\tilde{\mathbb{R}}^3]$ is *not* a quantum double. It is, however, still an example of our more general double crossproduct. Hence there is a canonical action on the position space algebra $\mathbb{C}[\mathbb{R}^3]$. It is the local spacetime in the model and we see that it is flat. Explicitly, the actions of the Lie algebra generators on scalar functions $f(\vec{x})$ on position space are defined by

$$P_a = -\iota \frac{\partial}{\partial x_a}, \quad J_a = -\iota \epsilon_{abc} x_b \frac{\partial}{\partial x_c}. \quad (25)$$

The physics which this theory describes has a model spacetime flat \mathbb{R}^3 , which means that in each patch of $\Sigma \times \mathbb{R}$ the motion is that of a free particle on \mathbb{R}^3 . There can still be a nontrivial e_3 connection but this is now everywhere flat regardless of the matter but with nontrivial transitions between patches, i.e. the particles respond to the background geometry but they do not act as sources for it. In short, the model is the quantum theory of a particle on a flat background, possibly nontrivial.

The semidual model with flipped conventions is given by

$$E_3 = SU_2 \ltimes \tilde{\mathbb{R}}^3,$$

where $\tilde{\mathbb{R}}^3$ has zero Lie bracket and zero Poisson bracket, which we identify with su_2 as a vector space. Its enveloping algebra is $U(su_2) \bowtie U(su_2) = U(su_2) \bowtie \mathbb{C}[\mathbb{R}^3]$ and acts naturally on the momentum coordinates $\mathbb{C}[\tilde{\mathbb{R}}^3]$. Clearly we can Fourier transform from functions on \mathbb{R}^3 to functions on $\tilde{\mathbb{R}}^3$ and back and thereby convert a construction in one model to one or the other where it will have a different interpretation. The algebraic structure, however, is self-dual under semidualization.

3.2. $D(U(su_2))$ —quantum gravity without cosmological constant (spin spacetime)

Next we take SU_2 with its zero Lie cobracket and su_2^* the dual Lie bialgebra, which means with the zero Lie bracket and Kirillov–Kostant Lie cobracket. The classical Poisson Lie group is the double $d(SU_2) = SU_2 \ltimes su_2^* = E_3$ again but this time with a non-trivial Poisson bracket. Its quantization is the quantum coordinate algebra of the quantum symmetry group $D(U(su_2)) = U(su_2) \bowtie \mathbb{C}[SU_2]$, where $\mathbb{C}[SU_2]$ is the coordinate algebra on the momentum space SU_2 and is described by a matrix of generators t^i_j dually paired with generators J_a of $U(su_2)$ by $\langle t^i_j, J_a \rangle = \frac{1}{2} \sigma_a^i_j$. Here σ_a are the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (26)$$

We describe the quantum double here in an algebraic form and with a parameter λ that expresses the ‘flattening’ of the momentum space SU_2 to \mathbb{R}^3 as $\lambda \rightarrow 0$. In the context of 3D quantum gravity one should take $\lambda = 1/m_p$. The algebraic quantum double then has generators J_a of su_2 and generators t^i_j of the coordinate algebra of SU_2 with relations

$$[t^i_j, J_a] = \frac{1}{2} (\sigma_a^i_l t^l_j - t^i_l \sigma_a^l_j), \quad [J_a, J_b] = \iota \epsilon_{abc} J_c$$

$$\Delta J_a = J_a \otimes 1 + 1 \otimes J_a, \quad \Delta t^i_j = t^i_l \otimes t^l_j.$$

We now change variables from t^i_j to $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ defined via

$$t^i_j = \mathcal{P}_0 \delta^i_j + \iota \frac{\lambda}{2} \mathcal{P}_c \sigma_c^i_j = \begin{pmatrix} \mathcal{P}_0 + \iota \frac{\lambda}{2} \mathcal{P}_3 & \iota \frac{\lambda}{2} (\mathcal{P}_1 - \iota \mathcal{P}_2) \\ \iota \frac{\lambda}{2} (\mathcal{P}_1 + \iota \mathcal{P}_2) & \mathcal{P}_0 - \iota \frac{\lambda}{2} \mathcal{P}_3 \end{pmatrix}.$$

The structure in terms of the new generators is

$$\mathcal{P}_0^2 + \frac{\lambda^2}{4} \vec{\mathcal{P}}^2 = 1, \quad [\mathcal{P}_0, J_a] = 0, \quad [\mathcal{P}_a, J_b] = \iota \epsilon_{abc} \mathcal{P}_c,$$

$$\Delta \mathcal{P}_0 = \mathcal{P}_0 \otimes \mathcal{P}_0 - \frac{\lambda^2}{4} \mathcal{P}_a \otimes \mathcal{P}_a, \quad \Delta \mathcal{P}_a = \mathcal{P}_a \otimes 1 + 1 \otimes \mathcal{P}_a - \frac{\lambda}{2} \epsilon_{abc} \mathcal{P}_b \otimes \mathcal{P}_c,$$

where the $\det t = 1$ relation appears now as the sphere relation for SU_2 as a 3-sphere in \mathbb{R}^4 , with $\vec{\mathcal{P}}$ the local coordinates of a patch of SU_2 containing the group identity. Here \mathcal{P}_a are regarded as the free variables valid for $|\vec{\mathcal{P}}| \leq 2/\lambda$ and $\mathcal{P}_0 = \sqrt{1 - \frac{\lambda^2}{4} \vec{\mathcal{P}}^2}$ in this patch. There

is another patch covering the lower half with $\mathcal{P}_0 \leq 0$. In either patch, we see that SU_2 as momentum space for this model is a curved version of \mathbb{R}^3 obtained in the limit $\lambda \rightarrow 0$. Note that the two patches above are not open sets, one should really use open patches and a third patch around the equator to see the limit topologically.

We have a canonical action of the quantum double on $U(su_2)$ which means on the flat but noncommutative spacetime algebra

$$[x_a, x_b] = \iota\lambda\epsilon_{abc}x_c, \tag{27}$$

where we recall that λ is $1/m_p$ i.e. proportional to the Planck length l_p in the context of 3D gravity (3). This is the enveloping algebra $U(su_2)$ with rescaled generators. The action of the quantum double on x_a is

$$J_a \triangleright x_b = \iota\epsilon_{abc}x_c, \quad \mathcal{P}_0 \triangleright x_a = x_a, \quad \mathcal{P}_a \triangleright x_b = \iota\delta_{ab},$$

see [3].

Finally, the $\vec{\mathcal{P}}$ coordinate system on momentum space SU_2 can be replaced by a local coordinate system \vec{p} valid near the group identity. Here an element of SU_2 is written as $e^{\frac{1}{2}\lambda\vec{p}\cdot\vec{\sigma}}$ in terms of a vector of Pauli matrices and valid for $|p| < 2\pi/\lambda$. The relation between the two coordinate systems is

$$\mathcal{P}_a = p_a \frac{\sin(\lambda|\vec{p}|/2)}{\lambda|\vec{p}|/2}, \quad \mathcal{P}_0 = \cos(\lambda|\vec{p}|/2).$$

Note that this second ‘Lie algebra’ coordinate system is degenerate at $|\vec{p}| = 2\pi/\lambda$ as all directions of \vec{p} then lead to the same point $-1 \in SU_2$. The noncommutative geometry of the model can be considerably developed [19, 35]. In particular, in any reasonable completion of the position coordinate algebra to include exponentials, the elements $\zeta = e^{\iota\vec{p}\cdot\vec{x}}$ with $|\vec{p}| = 2\pi/\lambda$ are non-trivial plane waves (of momentum -1) obeying $\zeta^2 = 1$ [19]. This means that noncommutative spacetime is a kind of double cover of noncommutative \mathbb{R}^3 in the same way that SU_2 is a double cover of SO_3 .

This model describes quantum gravity without cosmological constant in the sense that compared to the model of section 3.1 the particles at each puncture of Σ act as sources for the implicitly defined ‘connection’. This is achieved by switching on a finite m_p or nonzero Newton constant G . The model spacetime is noncommutative and the ‘connection’ is implicitly defined by its quantum group ‘holonomy’ so is in that sense ‘quantum’. It is actually the combination $l_p = \hbar/m_p$ that enters so one could view the model equivalently as switching on \hbar for fixed G . In this way, the theory describes quantum gravity coupled to the sources in contrast to section 3.1 where the background geometry on $\Sigma \times \mathbb{R}$ remains classical and unaffected by the sources.

3.3. \widetilde{SO}_4 —free particle with positive cosmological constant (SU_2 spacetime) as semidual of quantum gravity without cosmological constant

Next we apply the semidualization construction to the previous quantum double spin model. Due to our analysis for any quantum double we obtain, in the present case, the quantum group

$$U(su_2) \bowtie U(su_2)^{\text{cop}} \cong U(su_2) \otimes U(su_2)^{\text{cop}} = U(su_2 \oplus su_2) = U(so_4),$$

which is actually a classical enveloping algebra, acting covariantly on the classical position algebra $\mathbb{C}[SU_2]$ by left and right translations. Note that in terms of the generators of rotations and ‘translations’ on the left we have commutation relations

$$[J_a, J_b] = \iota\epsilon_{abc}J_c, \quad [J_a, P_b] = \iota\epsilon_{abc}P_c, \quad [P_a, P_b] = \iota\lambda\epsilon_{abc}P_c, \tag{28}$$

where in this model $\lambda = 1/l_c$. Its action on $\mathbb{C}[SU_2]$ is with J_a acting as the vector fields for conjugation and P_a acting as the vector fields for right translation. We can choose coordinates

on SU_2 with parameter λ as in section 3.2, just now SU_2 is position space, with P_a becoming usual differentiation on flat \mathbb{R}^3 as $\lambda \rightarrow 0$. This represents a fairly perverse but physical way of thinking about left and right translations on SU_2 which we will develop further.

We see that the semidual of our flat but noncommutative spacetime and quantum gravity system is a system with curved but classical model spacetime SU_2 . At the group level the euclidean group is now deformed to $SU_2 \bowtie SU_2$ which is isomorphic to $SU_2 \times SU_2$ and we view this as a double cover \widetilde{SO}_4 . In terms of the notation (A.2), the left copy of SU_2 acts by the vector fields ξ^L and the right copy by the vector fields ξ^R on functions of the position space SU_2 . The theory deforms the flat model of section 3.1 in now describing a quantum particle on a classical background with curvature (due to the cosmological constant) but insensitive to the sources. The motion looks locally like free motion on 3-spheres in each patch of $\Sigma \times \mathbb{R}$ with \widetilde{SO}_4 transitions.

This model is not self-dual as it is clearly very far from the previous model in section 3.2. Thus, a construction in quantum gravity but without cosmological constant maps over under semidualization to a construction on classical SU_2 . In physical terms of the original model this SU_2 is the curved momentum space. In the dual theory it is the curved position space. Conversely, a classical particle in the semidual theory means a particle on SU_2 with $SU_2 \times SU_2$ isometry group. It maps back to something else in the noncommutative spacetime of the quantum gravity model. We shall give details of both sides in section 4.

3.4. $\widetilde{SO}_{1,3}$ —free particle with negative cosmological constant (hyperbolic spacetime)

Here we take, in place of E_3 , the classical group

$$SL_2(\mathbb{C}) = SU_2 \bowtie SU_2^*$$

but with the zero Poisson bracket. Its structure is a double crossproduct of SU_2 and a certain solvable group $SU_2^* = \mathbb{R}^2 \bowtie \mathbb{R}$ occurring in the Iwasawa factorization. Each element of $SL_2(\mathbb{C})$ may be uniquely factorized in the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \begin{pmatrix} w & z \\ 0 & w^{-1} \end{pmatrix}, \quad |x|^2 + |y|^2 = 1, \quad w > 0, \quad x, y, z \in \mathbb{C}.$$

Such a matrix is in $SL_2(\mathbb{C})$ and, conversely, given a matrix as on the left, we define

$$w = \sqrt{|a|^2 + |c|^2}, \quad x = w^{-1}a, \quad y = w^{-1}c, \quad z = w^{-1}(\bar{a}b + \bar{c}d).$$

Note that the group SU_2^* and the Iwasawa factorization can be understood in Poisson Lie terms [20]. Thus, the former is the dual of SU_2 as a Poisson Lie group with its Drinfeld–Sklyanin Poisson bracket and $SL_2(\mathbb{C})$ is the classical double of SU_2 as a Poisson Lie group, but in the present model we use only the resulting $SL_2(\mathbb{C})$ group and factorization structure, taking it with zero Poisson structure.

There is a canonical right action of $SL_2(\mathbb{C})$ from the classical group double crossproduct theory on the set SU_2^* as a classical but curved position space,

$$b \triangleleft (g \bowtie a) = (b \triangleleft g) \cdot a.$$

Using the above we can compute \triangleleft explicitly as

$$\begin{pmatrix} w & z \\ 0 & w^{-1} \end{pmatrix} \triangleleft \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} = \begin{pmatrix} w' & z' \\ 0 & w'^{-1} \end{pmatrix}$$

$$w' = \sqrt{w^{-2}|y|^2 + |wx + zy|^2}, \quad w'z' = (w\bar{x} + \bar{z}\bar{y})(z\bar{x} - w\bar{y}) + w^{-2}\bar{x}\bar{y}.$$

In this way, $SL_2(\mathbb{C})$ becomes the isometry group of this position space with its natural hyperbolic metric, and the double crossproduct structure exhibits it explicitly as a curved

position space analogue of the euclidean group of motions. SU_2 acts as ‘deformed rotations’ \triangleleft and ‘deformed momentum’ SU_2^* acts by group right translation. In its internal structure, SU_2 also acts on momentum by the same deformed action \triangleleft but as $SL_2(\mathbb{C})$ is not a semidirect product, there is also a back-reaction

$$\begin{pmatrix} w & z \\ 0 & w^{-1} \end{pmatrix} \triangleright \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} = w'^{-1} \begin{pmatrix} wx + zy & -w^{-1}\bar{y} \\ w^{-1}y & w\bar{x} + \bar{z}\bar{y} \end{pmatrix}$$

of momentum on rotations as a result of the curved space.

At the algebraic level we have a left action of $U(sl_2) = U(d(su_2)) = U(su_2) \triangleright U(su_2^*)$ on $\mathbb{C}[SU_2^*]$ as the commutative coordinate algebra of functions on the classical but curved position space SU_2^* . Explicitly, the generators of sl_2 as isometry Lie algebra are J_a as usual for rotations and P_a , say, for ‘translations’, with nonzero commutation relations

$$\begin{aligned} [J_a, J_b] &= \iota \epsilon_{abc} J_c, & [P_3, P_i] &= \iota \lambda P_i, \\ [J_a, P_b] &= \iota \epsilon_{abc} P_c + \iota \lambda \delta_{b3} J_a - \iota \lambda \delta_{ab} J_3, \end{aligned} \tag{29}$$

where $i = 1, 2$ and $\lambda = 1/l_c$ in this model. The parameter ensures that we recover e_3 as $\lambda \rightarrow 0$. Note that the quantum group in this example is a classical enveloping algebra and therefore is *not* a quantum double of anything. Rather, it is the exponentiation of a classical Lie algebra double with zero cobracket in line with what we have explained above.

Finally, since the above action of $SL_2(\mathbb{C})$ on SU_2^* is quite complicated, it can be helpful to write the latter in a more suitable form as the upper half of the two-sheeted hyperboloid in 3+1 Minkowski space. This is also topologically \mathbb{R}^3 and comes with its own natural hyperbolic metric induced from the inclusion. The group structure is not manifest in this description, however. To give the change of coordinates we write elements of Minkowski space as 2×2 Hermitian matrices x , with determinant 1 for the unit hyperboloid. An element $g \in SL_2(\mathbb{C})$ acts on such a matrix via $x \mapsto g^\dagger x g$. We identify the unit matrix (the point $(1, 0, 0, 0)$ in usual time–space form) here with the unit matrix of SU_2^* . Our factorization of $SL_2(\mathbb{C})$ is exactly into the subgroup SU_2 of spatial rotations that leaves this point invariant and the subgroup of boosts which is SU_2^* and acts by (in the conventions above) right multiplication. Thus a general point of SU_2^* corresponds to a 2×2 Hermitian matrix in the upper half hyperboloid by

$$\begin{pmatrix} w & z \\ 0 & w^{-1} \end{pmatrix} \leftrightarrow \begin{pmatrix} w & z \\ 0 & w^{-1} \end{pmatrix}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w & z \\ 0 & w^{-1} \end{pmatrix} = \begin{pmatrix} w^2 & wz \\ w\bar{z} & w^{-2} + z\bar{z} \end{pmatrix}.$$

One can coordinatize SU_2^* with coordinates of length dimension in a variety of ways, for example

$$w = 1 + \lambda \mathcal{X}_3, \quad z = \lambda(\mathcal{X}_1 + \iota \mathcal{X}_2), \quad \mathcal{X}_3 > -\frac{1}{\lambda}.$$

Then the group structure appears as a modified addition law of \mathbb{R}^3 , see [12]. Equipped with a compatible Riemannian metric, hyperbolic space is a curved deformation of \mathbb{R}^3 , becoming flat in the limit $\lambda \rightarrow 0$. One also has Lie algebra coordinates x_a with matrix $e^{\iota \vec{x} \cdot \vec{p}}$ for certain matrices ρ_a . The exponential map here is a bijection with \mathbb{R}^3 .

The model has a similar physical interpretation to that of section 3.3, i.e. quantum particles on a classical background with curvature (due to the presence of a cosmological constant) but uncoupled to the sources. The difference is that the motion is locally described by motion on hyperbolic 3-space with $SL_2(\mathbb{C})$ transitions between patches.

3.5. $U(su_2) \bowtie \mathbb{C}[SU_2^*]$ —semidual of free particle in hyperbolic space (bicrossproduct spacetime)

Next, we apply the semidualization construction to the preceding model with spacetime curvature. Once again, this interchanges the role of position and momentum at a Hopf-algebraic level. Hence space becomes the flat but noncommutative ‘bicrossproduct spacetime’ whose coordinate algebra is the enveloping algebra $U(su_2^*)$, i.e. with non-zero brackets

$$[x_i, x_3] = \iota \lambda x_i \tag{30}$$

for $i = 1, 2$, where the deformation parameter λ should be interpreted as $1/m_p$ in this model. Meanwhile, rotations remain unchanged as SU_2 or $U(su_2)$ at the Hopf algebra level while the enveloping algebra of momentum is the commutative algebra of functions on SU_2^* . This is the bicrossproduct euclidean quantum group. Its dual can be viewed as quantizing the bicrossproduct Poisson Lie group $SU_2 \bowtie su_2$ where su_2 is an additive group, with a certain bicrossproduct Poisson Lie structure [18]. The classical group here is once again E_3 but with a different Poisson Lie group structure than in some of the above models.

To give details, in order to have all quantum groups left-acting, we again flip conventions to a conjugate factorization $SL_2(\mathbb{C}) = SU_2^* \cdot SU_2$, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} w & 0 \\ z & w^{-1} \end{pmatrix} \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix}, \quad |x|^2 + |y|^2 = 1, \quad w > 0, \quad x, y, z \in \mathbb{C},$$

$$w = \sqrt{|a|^2 + |b|^2}, \quad x = w^{-1}a, \quad y = w^{-1}b, \quad z = w^{-1}(\bar{a}c + \bar{b}d).$$

This implies a Hopf algebra factorization $U(sl_2) = U(su_2^*) \bowtie U(su_2)$ as a version of the classical cosmological model above. Semidualization using the A -version of the theory (in the terminology of section 2.6) then gives a new Hopf algebra $U(su_2) \bowtie \mathbb{C}[SU_2^*]$ which acts canonically on $U(su_2^*)$. This can be computed explicitly cf [12, 18]

$$[J_a, J_b] = \iota \epsilon_{abc} J_c, \quad [P_a, J_3] = \iota \epsilon_{a3c} P_c, \quad [P_3, J_a] = \iota \epsilon_{3ab} P_b$$

$$[P_a, J_b] = \frac{\iota}{2} \epsilon_{ab3} \left(\frac{1 - e^{-2\lambda P_3}}{\lambda} - \lambda(P_1^2 + P_2^2) \right) + \iota \lambda \epsilon_{ac3} P_b P_c,$$

giving a nonlinear action of su_2 on the manifold of SU_2^* . This manifold can be naturally identified with hyperbolic space, as explained at the end of section 3.4. Meanwhile, as indicated in the bicrossproduct notation, the coalgebra also has a semidirect form

$$\Delta J_i = J_i \otimes 1 + e^{-\lambda P_3} \otimes J_i + \lambda P_i \otimes J_3, \quad \Delta P_i = P_i \otimes 1 + e^{-\lambda P_3} \otimes P_i$$

for $i = 1, 2$ and the usual additive ones for P_3, J_3 .

The action of this quantum group on the bicrossproduct position algebra $U(su_2^*)$ is

$$J_a \triangleright x_b = \iota \epsilon_{abc} x_c, \quad P_a \triangleright : f(x) :=: \frac{\partial}{\partial x_a} f(x) :$$

where $: :$ denotes normal ordering of an ordinary polynomial with x_3 to the right.

3.6. $D(U_q(su_2))$ —quantum gravity with cosmological constant (q -hyperbolic spacetime $B_q[SU_2]$)

Finally, we can follow the same ideas but now in quantum gravity with cosmological constant, where there are no classical groups or spaces on either side of the semidualization. We are actually going to give some different versions algebraically equivalent when $q \neq 1$ by ‘purely

quantum' isomorphisms. Note that for the quantum group $U_q(su_2)$ we use the standard generators H, X_{\pm} so that

$$q^{\frac{H}{2}} X_{\pm} q^{-\frac{H}{2}} = q^{\pm 1} X_{\pm}, \quad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad (31)$$

as well as

$$\Delta q^{\pm \frac{H}{2}} = q^{\pm \frac{H}{2}} \otimes q^{\pm \frac{H}{2}}, \quad \Delta X_{\pm} = q^{-\frac{H}{2}} \otimes X_{\pm} + X_{\pm} \otimes q^{\frac{H}{2}}.$$

The real form here is defined by $H^* = H$ and $X_{\pm}^* = X_{\mp}$ at least for real q (the root of unity case is more subtle). For its dual $\mathbb{C}_q[SU_2]$ we use a matrix of generators $t^i_j = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with its usual relations

$$\begin{aligned} ba &= qab, & bc &= cb, & bd &= q^{-1}db, \\ ca &= qac, & cd &= q^{-1}dc, & da &= ad + (q - q^{-1})bc \end{aligned}$$

and matrix form of coproduct. The real form is given by $a^* = d, b^* = -q^{-1}c$ for q real.

For our first version in figure 1, the form suggested by the classical geometry is the quantum double viewed as

$$U_q(so_{1,3}) = U_q(su_2) \bowtie U_q(su_2^*),$$

where $U_q(su_2^*) \cong \mathbb{C}_q[SU_2]^{\text{op}}$ with new generators ξ, x and y defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^{\xi} & \lambda y \\ \lambda x & q^{-\xi}(1 + q\lambda^2 xy) \end{pmatrix}, \quad \lambda = q^{-1} - q, \quad (32)$$

and relations and coproduct that the reader can translate. For example, the relations here are

$$[\xi, x] = x, \quad [\xi, y] = y, \quad [x, y] = 0, \quad (33)$$

so as an algebra it is in fact $U(su_2^*)$, undeformed. This is the 'purely quantum isomorphism' on the lower left in figure 1, valid for $q \neq 1$. Note that in this model the small deformation parameter $\lambda \approx 2/(m_p l_c)$ is, like q , dimensionless. The quantum double in this form is the dual of the quantum group quantizing $su_2 \bowtie su_2^*$ with its classical double Poisson Lie group structure. There is a canonical action on $U_q(su_2)^{\text{cop}} = U_{q^{-1}}(su_2)$ with generators h, x_{\pm} , say (to distinguish from the previous ones) and relations with inverted q . This could serve as a definition of $\mathbb{C}_q[SU_2^*]$ as a noncommutative space with generators w and z defined via

$$\begin{pmatrix} w & z \\ 0 & w^{-1} \end{pmatrix} = \begin{pmatrix} q^{\frac{h}{2}} & q^{-\frac{1}{2}} \lambda x_- \\ 0 & q^{-\frac{h}{2}} \end{pmatrix},$$

a matrix form of coalgebra and relations that the reader can translate from those of $U_q(su_2)$. One needs the complex conjugate as an additional generator z^* of $\mathbb{C}_q[SU_2^*]$ to complete this to a $*$ -algebra along with $w^* = w$ as a real generator. This version of the model is a q -deformation of the free particle on hyperbolic spacetime (the middle left model of figure 1, section 3.4), with q -deformation the introduction of finite m_p or the 'switching on' of mutual gravitational interaction via the Newton constant G .

Next, as in the classical case, it is natural to define this q -hyperbolic space as the unit mass hyperboloid of q -Minkowski space. The necessary q -Minkowski space is defined as the coordinate algebra $B_q[M_2]$ of the space of 2×2 braided Hermitian matrices [11, 12]

$$\beta\alpha = q^2\alpha\beta, \quad \gamma\alpha = q^{-2}\alpha\gamma, \quad \delta\alpha = \alpha\delta,$$

$$[\beta, \gamma] = (1 - q^{-2})\alpha(\delta - \alpha), \quad [\delta, \beta] = (1 - q^{-2})\alpha\beta, \quad [\gamma, \delta] = (1 - q^{-2})\gamma\alpha,$$

$$\begin{aligned} \underline{\Delta} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \\ \underline{\epsilon} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}. \end{aligned}$$

The coproduct here extends to products with braid statistics, much as for super-matrices but with Bose–Fermi statistics replaced by a braiding matrix. If we quotient by the braided-determinant relation $\alpha\delta - q^2\gamma\beta = 1$ we have the unit hyperboloid in q -Minkowski space, which is the coordinate algebra of the braided group $B_q[SU_2]$. The q -determinant otherwise defines a q -metric. When $q \neq 1$ this algebra is more or less isomorphic to $U_q(su_2)$ as required by means of the ‘quantum Killing form’, as

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} w & z \\ 0 & w^{-1} \end{pmatrix}^* \begin{pmatrix} w & z \\ 0 & w^{-1} \end{pmatrix} = \begin{pmatrix} q^h & q^{-\frac{1}{2}}\lambda q^{\frac{h}{2}}x_- \\ q^{-\frac{1}{2}}\lambda x_+ q^{\frac{h}{2}} & q^{-h} + q^{-1}\lambda^2 x_+ x_- \end{pmatrix}$$

in terms of our previous identification. This quantum Killing form can also be viewed more categorically as essentially an isomorphism between the braided enveloping algebra $BU_q(su_2)$ (which has the same algebra as $U_q(su_2)$) and its dual which is the braided function algebra $B_q[SU_2]$.

For our second version of $D(U_q(su_2))$ we come from the quantum double construction rather than the classical version. So we work with $D(U_q(su_2)) = U_q(su_2) \bowtie \mathbb{C}_q[SU_2]^{\text{op}}$ acting likewise on $U_q(su_2)^{\text{cop}}$ viewed as $\mathbb{C}_q[SU_2^*]$ or by preference as $B_q[SU_2]$. Moreover, it turns out to be very natural to replace $\mathbb{C}_q[SU_2]^{\text{op}}$ in the quantum double by another copy of $B_q[SU_2]$ with matrix generators u^i_j , say. Then one finds

$$D(U_q(su_2)) \cong U_q(su_2) \bowtie B_q[SU_2],$$

which is then a semidirect product as an algebra and as a coalgebra, called the ‘bosonization’ of $B_q[SU_2]$ [12]. Here $U_q(su_2)$ acts on $B_q[SU_2]$ both as spacetime and as rotations by the quantum coadjoint action. This form of the quantum double expresses the model as a q -deformation of quantum gravity without cosmological constant in section 3.2, i.e. as purely introducing the cosmological constant.

Finally, using this braided theory we are able better to understand our first version, as a third formulation of the quantum double

$$U_q(so_{1,3}) = U_q(su_2) \blacktriangleright U_q(su_2),$$

which as an algebra is the tensor product one. This describes $U_q(so_{1,3})$ as a complexification of $U_q(su_2)$ and a further ‘twisting’ of the coproduct. This form of the quantum double follows from the $U_q(su_2) \bowtie BU_q(su_2)$ form (using the quantum Killing form isomorphism above) and the fact that the semidirect product by the quantum adjoint action used for the algebra structure can then be unravelled to a tensor product. This explains our two points of view of the model as shown on the left side of the lower block in figure 1. They are isomorphic provided $q \neq 1$, a ‘purely quantum’ phenomenon.

3.7. $U_q(so_4)$ —semidual of quantum gravity with cosmological constant ($\mathbb{C}_q[SU_2]$ spacetime) and self-duality

The semidual of the preceding quantum double model has quantum group $U_q(su_2) \otimes U_q(su_2)^{\text{cop}} = U_q(so_4)$ acting on the q -deformed space $\mathbb{C}_q[SU_2]^{\text{op}}$. The action here is by left and right differentials, i.e. by the coproduct of $\mathbb{C}_q[SU_2]$ viewed as a left or right coaction and evaluated against the two copies of $U_q(su_2)$. This version of the model

exactly q -deforms the semidual of quantum gravity without cosmological constant based on \widetilde{SO}_4 acting on SU_2 , i.e. it q -deforms the free particle on SU_2 with cosmological constant (the upper right of figure 1) with q -deformation introducing mutual gravitational interactions via finite m_p or non-zero Newton constant G .

Theorem 3.1. *For generic $q \neq 1$, or for the reduced theory at q a root of unity, quantum gravity with cosmological constant as above is self-dual up to an algebraic equivalence under semidualization. The algebraic equivalence is given by a quantum Wick rotation [36] or ‘transmutation’ from $\mathbb{C}_q[SU_2]^{\text{op}}$ to $B_q[SU_2]$ as spacetime algebra and a Drinfeld twist from $U_q(su_2) \blacktriangleright U_q(su_2)$ to $U_q(su_2) \otimes U_q(su_2)^{\text{cop}}$ as q -isometry group.*

The Drinfeld twist needed is the composition of two; one to convert $U_q(su_2)^{\text{cop}}$ to $U_q(su_2)$ and the second to convert $U_q(su_2) \otimes U_q(su_2)$ over to $U_q(su_2) \blacktriangleright U_q(su_2)$. The Drinfeld twist here conjugates the coproduct by a Hopf-cocycle. Its key feature is that *it does not change the category of modules up to a formal equivalence*. More precisely, since in this form the algebras of the two quantum groups are the same, their category of modules has the same objects. Tensor products of two modules depend on the coproduct and these are related by a twisting cocycle obtained from the braiding or ‘universal R -matrix’ of $U_q(su_2)$ (so the tensor products are nontrivially isomorphic by this cocycle). Details were developed by one of the authors in the early 1990s and are in [12] and elsewhere. To do this rigorously, however, one has to look at the convergence of power series or work not over \mathbb{C} but over the ring of formal power series in the deformation parameter. The reader can also say quite rightly that the categories of modules of $U(so_{1,3})$ and $U(so_4)$ are quite different and cannot possibly coincide. Indeed, the only difference in the classical case is the $*$ -structure or unitarity constraint. However, in the q -deformed theory $U_q(so_{1,3})$ and $U_q(so_4)$ are different even as Hopf algebras and it is at this algebraic level that we have the equivalence (i.e., not respecting the $*$ -structures of the quantum groups, which are not equivalent). Also, in physical terms the situation is actually *more* precise when $\Lambda > 0$ i.e. when q is a complex number of modulus 1, and we look at the truncated theory at q a root of unity. In this case, we must use finite ‘reduced’ versions of all our algebras and have exact isomorphisms. Some theory of $\mathbb{C}_q[SU_2]$ at q a root of unity is in [37].

The specific twists here also have a deep braided category interpretation which is the origin of the term ‘transmutation’. This theory converts ordinary quantum groups such as $\mathbb{C}_q[SU_2]$ into braided ones such as $B_q[SU_2]$ but in such a way that all of the theory has braided parallels. In particular, there is also a braided version $BU_q(su_2)$ of $U_q(su_2)$ and the (essentially) isomorphism $B_q[SU_2] \cong BU_q(su_2)$ has a categorical origin as braided selfduality of such ‘factorizable’ quantum groups. Because of it, the braided Fourier transform becomes an operator $B_q[SU_2] \rightarrow B_q[SU_2]$ which, together with left multiplication by the ribbon element generates a representation of the mapping class group $PSL(2, \mathbb{Z})$ [38]. This representation is at the heart of the three-manifold invariant corresponding to the quantum group $U_q(su_2)$. The same applied to $D(U_q(su_2))$ is at the heart of the Turaev–Viro invariant, i.e. of the solution of this part of 3D quantum gravity with cosmological constant. Moreover, because the quantum gravity theory with point sources is controlled essentially by attaching representations at the marked points of the Riemann surface as explained in section 2, the semidual $U_q(so_4)$ theory has in some sense the same physical content up to the mentioned (but non-trivial) isomorphisms.

Finally, using the dual of the ‘purely quantum isomorphism’ (32) we arrive at the other version on the lower right of figure 1 with quantum isometry group $U_q(su_2) \blacktriangleright \mathbb{C}_q[SU_2^*]$, isomorphic when $q \neq 1$. We arrive this time at the q -deformation of the bicrossproduct model

of section 3.5, so q -deformation is now interpreted once again as introducing the cosmological constant.

3.8. Degenerations between the models

As indicated in figure 1 the various models as well as being related by semidualization horizontally are related vertically by ‘deformation’ going downwards or, going the other way, by degeneration. In this subsection, we explain these degeneration maps between the models. The key observation is that the q -deformed models in sections 3.6 and 3.7 (the bottom of figure 1) have only one parameter q which is dimensionless. However, the limit $q \rightarrow 1$ can be taken in different ways according to how the generators are scaled and this gives various degenerations.

We start at the bottom left of figure 1, the dimensionless model of section 3.6. For the rotational part of the isometry quantum group there is no problem and we set $U_q(su_2) \mapsto U(su_2)$ as $q \rightarrow 1$. However, $U_q(su_2)$ is *also* the quantum spacetime algebra in the model at the bottom left of figure 1, and here we must be more careful to write

$$H = 2m_p x_3, \quad x_{\pm} = m_p(x_1 \pm ix_2)$$

and then take the limit $l_c \rightarrow \infty$ in relations (31) of $U_q(su_2)$. We then obtain the spin model spacetime of section 3.2 (the upper left of figure 1) with relations (27) for x_a .

On the other hand, we can make use of the ‘purely quantum isomorphism’ (32) and set

$$\xi = -il_c P_3, \quad x = l_c(P_1 + iP_2), \quad y = -x^*$$

and then take the limit $m_p \rightarrow \infty$. We then obtain the momentum sector of the classical hyperbolic spacetime model of section 3.4 (left middle of figure 1): relations (33) become the relations for P_a in (29).

Similarly on the right-hand side of figure 1 starting at the bottom in the dimensionless q -deformed theory of section 3.7, we can set $q \rightarrow 1$ *after* identifying the q -bicrossproduct as $U_q(su_2) \otimes U_q(su_2)^{\text{cop}}$. By setting $q \rightarrow 1$ here and for the spacetime $\mathbb{C}_q[SU_2]^{\text{op}}$ we obtain the particle on a 3-sphere in section 3.3 (upper right in figure 1). Finally, on the other side of the ‘purely quantum isomorphism’ (32) we can write

$$\xi = im_p x_3, \quad x = m_p(x_1 + ix_2), \quad y = -x^*,$$

and then take the limit $l_c \rightarrow \infty$ to obtain the bicrossproduct spacetime model of section 3.5 (right middle of figure 1): relations (33) turn into the spacetime algebra (30) for x_a .

We have described here the degenerations at the level of spacetime and isometry algebras. The same applies when one looks deeper into the noncommutative differential geometry of the models. For example, the standard 4D bicovariant differential calculus on $C_q[SU_2]^{\text{op}}$ at bottom right degenerates to a 4D quantum-isometry covariant differential calculus on the bicrossproduct spacetime. This will be given in detail elsewhere. The final degenerations to the E_3 model of section 3.1 at the top of the figure are obvious as the remaining m_p or l_c parameter is set to infinity.

4. Physics of semiduality from spin spacetime to classical SU_2

So far we have been describing our models in terms of the algebraic structure of isometry (quantum) groups and their relation by semidualization. In this section we now look in detail at the physics in the sense of the irreducible representations in these models, concentrating on the upper part of our overview in figure 1. Our main motivation is to understand the physical interpretation of semiduality, using the strategy outlined in section 1: by studying

irreps of the isometry (quantum) groups and their semiduals, and using (quantum) Fourier transforms to switch from momentum to position representation within one model we are able to realize representations of mutually semidual (and generally non-equivalent) models in terms of functions on the *same* (possibly noncommutative) space. An additional motivation for studying irreps and their Fourier transforms comes from the role of quantum doubles in the construction of the Hilbert space (18) of 3D quantum gravity, where the irreps represent the one-particle contributions. While the literature on 3D quantum gravity has focussed on the momentum space picture of those irreps, the (noncommutative) position picture may provide insights into the interpretation of 3D quantum gravity in terms of noncommutative geometry.

We recall that semiduality maps the euclidean group E_3 to itself (but exchanges momenta and positions), whereas the quantum double $D(U(su_2))$ (spin model) is mapped to the universal enveloping algebra of so_4 (S^3 spacetime model). The case of E_3 is thus exceptional in that semiduality and Fourier transform coincide. We give the irreps of E_3 both in terms of vector-valued functions on momentum space satisfying a (multiplicative) spin constraint and in terms of vector-valued functions on position space satisfying a Dirac-type linear first-order wave equation; the two pictures are related via standard Fourier transform. This case is of course the well-known Wigner construction but we present it in a geometrical form that is suitable for deformation. For $D(U(su_2))$ or more precisely $D(SU_2)$ in a global formulation, the irreps are given precisely as a $1/m_p$ -deformation of the E_3 picture, both in terms of vector-valued functions on curved momentum space SU_2 satisfying a (multiplicative) group-valued spin constraint cf [5] and in terms of noncommutative wave equations on fuzzy \mathbb{R}^3 as in [3]. For so_4 the irreps are given in terms of vector-valued functions on curved position space SU_2 satisfying a linear first-order differential equation, and, after quantum Fourier transform, in terms of vector-valued functions on fuzzy momentum space satisfying an algebraic constraint. We show that this, too, is a deformation of E_3 , this time recovered as $l_c \rightarrow \infty$. Thus we obtain a precise dictionary between the physical pictures in the two non-trivial models. They are not equivalent, but are both deformations of the same pictures in the E_3 case.

4.1. Representations of E_3

We first recall some standard facts and notations for su_2 and its representations. We introduce a set of Hermitian generators t_a satisfying the standard commutation relations

$$[t_a, t_b] = \iota \epsilon_{abc} t_c$$

and given explicitly via $t_a = \sigma_a/2$ in terms of Pauli matrices. We will denote the $(2s + 1)$ -dimensional irreducible representation of the Lie algebra su_2 by ρ^s , where $s \in \frac{1}{2}(\mathbb{N} \cup \{0\})$. This has a lowest weight vector which we will denote by $|s, -s\rangle$, where $\rho^s(t_3)|s, -s\rangle = -s|s, -s\rangle$ in our conventions. For $s = 1$ it will be convenient to consider the Cartesian basis, where

$$\rho^1(t_a)_{bc} = -\iota \epsilon_{abc}, \tag{34}$$

and for $s = 1/2$ it will be convenient to use the defining Pauli matrix representation $\rho^{1/2}(t_a) = t_a = \frac{1}{2}\sigma_a$. We will also use the t_a basis to identify su_2 with \mathbb{R}^3 . However, all of our constructions are basis independent.

The euclidean group $E_3 = SU(2) \ltimes \mathbb{R}^3$ was covered in section 3.1 and we use the notations from there. In particular, recall that the translation part is identified with su_2^* , with generators denoted by P_a so that a finite translation is written as $a = -\iota a_b P_b$. According to the standard theory, irreps of E_3 are labelled by $SU(2)$ orbits in momentum space $(su_2^*)^*$ together with irreps of associated centralizers. Since $(su_2^*)^* = su_2$, momentum space is su_2 and we could use the basis $\{t_a\}$, but we need to be careful about normalization. As explained

in section 3.1 the dual basis $\{P_a^*\}$ may have a different normalization from that of $\{t_a\}$, which is fixed by the commutation relations, so we should allow

$$P_a^* = -\lambda t_a, \tag{35}$$

where λ is an arbitrary constant of dimension inverse mass. Thus we view su_2 as momentum space and denote elements as p , which we expand as

$$p = \iota p_a P_a^* = -\iota \lambda p_a t_a \tag{36}$$

if we wish to use an \mathbb{R}^3 notation. We should stress that the parameter λ only enters the discussion because we choose to work with the basis $\{t_a\}$ of momentum space; if we carried out the analysis entirely in terms of the basis $\{P_a^*\}$ this parameter would not be required.

The irreducible representations of E_3 are then labelled by adjoint SU_2 orbits i.e. by two-spheres $S_m^2 = \{v \lambda m t_3 v^{-1} \mid v \in SU_2\}$ in momentum space and irreducible unitary representations Π_s of associated stabilizers $N_m = \{g \in SU_2 \mid g \lambda m t_3 g^{-1} = \lambda m t_3\}$. Clearly $N_0 \simeq SU_2$ and $N_m \simeq U(1)$ for all other values of m and $s \in \frac{1}{2}(\mathbb{N} \cup \{0\})$. The parameters m and s are interpreted as (euclidean) mass and spin of a particle. In the generic case the carrier spaces for the irreducible representations are

$$V_{ms} = \{\psi : SU_2 \rightarrow \mathbb{C} \mid \psi(v e^{\alpha t_3}) e^{i s \alpha} \psi(v), \forall \alpha \in [0, 4\pi), \forall v \in SU_2\}, \tag{37}$$

whose elements also arise as sections of Dirac monopole bundles, and we therefore refer to them as monopole sections. An element $(g, a) \in E_3$ acts on a monopole section via

$$\pi_{ms}((g, a))\psi(v) = \exp(\iota m a (\text{Ad}_{g^{-1}v}(\iota P_3^*)))\psi(g^{-1}v). \tag{38}$$

If we introduce the su_2 element

$$p = \iota m v P_3^* v^{-1}, \tag{39}$$

the phase here could be written as

$$\exp(\iota \vec{a} \cdot \text{Ad}_{g^{-1}}(\vec{p}))$$

when both a and p are expanded in the mutually dual bases $\{-\iota P_a\}$ and $\{\iota P_a^*\}$. For $m = 0$ the centralizer representations are SU_2 representations. In the resulting finite-dimensional representations of E_3 , the translations act trivially. We are not interested in the finite-dimensional irreducible representations in the following.

Given $\psi \in V_{ms}$ define the map

$$\tilde{\phi} : S_m^2 \rightarrow \mathbb{C}^{2s+1}, \tag{40}$$

where S_m^2 is the 2-sphere in su_2 of radius λm , via

$$\tilde{\phi}(p) = \psi(v) \rho^s(v) |s, -s\rangle, \tag{41}$$

where p is related to v via (39). Clearly

$$\rho^s(v e^{\alpha t_3}) |s, -s\rangle = \rho^s(v) \rho^s(e^{\alpha t_3}) |s, -s\rangle = \rho^s(v) e^{-\iota \alpha s} |s, -s\rangle$$

which cancels the phase picked up by ψ under the right-multiplication by $e^{\alpha t_3}$. Hence $\tilde{\phi}$ only depends on $p \in S_m^2$ even though both $\rho^s(v)$ and ψ depend on v .

The map $\tilde{\phi}$ defined in (41) satisfies the constraint

$$(\rho^s(t_a) p_a + m s) \tilde{\phi} = 0. \tag{42}$$

To see this, write (39) as $p_a t_a = v m t_3 v^{-1}$ so that

$$\begin{aligned} \rho^s(t_a) p_a \tilde{\phi}(p) &= \rho^s(v m t_3 v^{-1}) \rho^s(v) \psi(v) |s, -s\rangle \\ &= \psi(v) \rho^s(v) m (-s) |s, -s\rangle \\ &= -m s \tilde{\phi}(p), \end{aligned}$$

as required. Conversely, any map $\tilde{\phi} : S_m^2 \rightarrow \mathbb{C}^{2s+1}$ satisfying this constraint can be written in the form (41) with $\psi \in V_{ms}$. Thus the field $\tilde{\phi}(p)$ is the monopole section corresponding to ψ but written ‘downstairs’ on the base S_m^2 of the monopole bundle as a function with values in a one-dimensional vector space within \mathbb{C}^{2s+1} that varies as we move about on the base, in other words as an element in a rank 1 projective module. There is an associated projection matrix at every point $p \in S_m$:

$$e(p) = \rho^s(v)|s, -s\rangle\langle s, -s|\rho^s(v^{-1}), \tag{43}$$

with (39) assumed, which projects any $\tilde{\phi}$ to a solution of our constraint, i.e. down to the irreducible representation. Note that for $s = 1/2$ we have

$$e(p) = \frac{1}{2} + \frac{t_a p_a}{m}, \tag{44}$$

while for other spins the relationship is more complicated.

To obtain a unified description of all (infinite-dimensional) irreducible representations we consider the union

$$\bigcup_{m \in \mathbb{R}^+} S_m^2 \simeq \mathbb{R}^3 \setminus \{0\}$$

and use the carrier space

$$W_s = \{\tilde{\phi} : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{C}^{2s+1}\}$$

as a starting point for the representation theory of E_3 . The subspaces

$$W_{ms} = \{\tilde{\phi} : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{C}^{2s+1} | (\rho^s(t_a)p_a + ms)\tilde{\phi} = 0\}$$

obtained by imposing the constraint are representation of E_3 . In order to obtain an irrep as before we may still need to impose an additional constraint

$$(p^2 - m^2)\tilde{\phi} = 0$$

although for spins 1/2, 1 this holds automatically. An element $(g, a) \in E_3$ acts via

$$\pi_{ms}((g, a))\tilde{\phi}(p) = e^{i a(\text{Ad}_{g^{-1}} p)} \rho^s(g)\tilde{\phi}(\text{Ad}_{g^{-1}} p),$$

which commutes with the constraint (42), as required. The angle in the phase here is again $\vec{a} \cdot \text{Ad}_{g^{-1}} \vec{p}$ in our chosen bases.

The advantage of working with the map $\tilde{\phi}$ in this way is that it is defined on a linear space. We can Fourier transform back to a field

$$\phi(x) = \frac{1}{\sqrt{(2\pi)^3}} \int d^3 p e^{i\vec{x} \cdot \vec{p}} \tilde{\phi}(p),$$

which turns the constraint (42) into the first-order differential equation

$$(t\rho^s(t_a)\partial_a - ms)\phi = 0. \tag{45}$$

For $s = \frac{1}{2}$ this is the Dirac equation

$$(t\sigma_a\partial_a - m)\phi = 0. \tag{46}$$

Applying the adjoint Dirac operator $i\sigma_a\partial_a + m$ we deduce

$$(\Delta + m^2)\phi = 0.$$

For $s = 1$ equation (45) takes the form

$$\nabla \times \phi = -m\phi, \tag{47}$$

where we used the Cartesian representation (34). Computing the divergence on both sides we deduce $\nabla \cdot \phi = 0$ and therefore, upon applying $\nabla \times$ to both sides of (47),

$$(\Delta + m^2)\phi = 0.$$

To sum up, we obtain irreducible representations of E_3 on the space of \mathbb{C}^{2s+1} valued ‘wavefunctions’ satisfying a first-order equation, which generalizes the Dirac equation

$$W_{ms} = \{\phi : \mathbb{R}^3 \rightarrow \mathbb{C}^{2s+1} \mid (t\rho^s(t_a)\partial_a - ms)\phi = 0\},$$

at least for spin 1/2 and 1. For higher spins one may need to supplement with the usual wave equation $(\Delta + m^2)\phi$ as for scalar fields. An element $(g, a) \in E_3$ acts on a wavefunction via

$$\pi_{ms}((g, a))\phi(\vec{x}) = \rho^s(g)\phi(\text{Ad}_{g^{-1}}(\vec{x}) - \vec{a}).$$

The infinitesimal generators P_a and J_a of translations and rotations act as

$$P_a = -t\frac{\partial}{\partial x_a}, \quad J_a = -t\epsilon_{abc}x_b\frac{\partial}{\partial x_c} + \rho^s(t_a), \tag{48}$$

so that $\vec{P} \cdot \vec{J} = -t\rho^s(t_a)\partial_a$ is the Casimir used in the definition of W_{ms} .

4.2. Representations of the quantum double $D(SU_2)$

We now look similarly at the particle states in the quantum double ‘spin model’ related to 3D quantum gravity without cosmological constant. We will view the quantum double here as a deformation of E_3 [3, 5] as we explained in section 3.2, with a parameter $\lambda = 8\pi/m_p$ in the quantum gravity application. Note that, with this choice for λ , relation (35) between rotation and dual translation generators is the identification (13) of J_a with P_a^* in terms of the non-degenerate symmetric form used in the Chern–Simons action for 3D gravity. As we shall see, the identification of P_a^* with J_a (or t_a), which was optional in the discussion of E_3 representations, is essential in the following discussion of quantum double representations. Our treatment is fully analogous to that of E_3 , including a physical interpretation as particles of some kind with mass and spin.

We start with some remarks about the relevant quantum double. Indeed, the required quantum double of a compact Lie group G has been studied in various publications and can be defined as a particular Hopf C^* -algebra. However, its formulation as such is quite technical and in practice one can take either a $*$ -algebraic approach in terms of generators and relations, or much as in physics one can work at the Lie algebra level in practice. Thus, $D(U(g)) = U(g) \bowtie \mathbb{C}[G]$ where $U(g)$ denotes the enveloping algebra of the Lie algebra of ‘rotations’ (in our application) and $\mathbb{C}[G]$ an algebra of coordinates in momentum space G . The semidirect product is by the right adjoint action and in the case of SU_2 the required structure was given in section 3.2 as derived in [3]. Note, however, that group elements do not themselves lie in $U(g)$ but in a completion, i.e. have to be approximated.

The more technical C^* approach makes use of a crossproduct $C^*(G) \bowtie C(G)$ of the group C^* -algebra and the C^* -algebra of continuous functions on G . The former is defined first by a convolution product of functions of compact support and then completed. A closely related approach [39] is to start with continuous functions on $G \times G$ with convolution on the first factor (note that we exchange the roles played by the two copies of G in order to match our conventions for the semidirect product group E_3). In these approaches one obtains eventually a Hopf C^* -algebra $D(G)$ but one still has to approximate the actual elements of the ‘rotation group’ copy of G since these would appear as δ -functions in the convolution algebra. If we allow these for purposes of writing simple formulae, we have multiplication \bullet , identity 1,

co-multiplication Δ , co-unit ϵ , antipode S and involution $*$ via

$$\begin{aligned} (F_1 \bullet F_2)(g, u) &:= \int_G F_1(z, zuz^{-1}) F_2(z^{-1}g, u) dz, \\ 1(g, u) &:= \delta_e(g), \\ (\Delta F)(g_1, u_1; g_2, u_2) &:= F(g_1, u_1 u_2) \delta_{g_1}(g_2). \\ \epsilon(F) &:= \int_G F(g, e) dg, \\ (SF)(g, u) &:= F(g^{-1}, g^{-1}u^{-1}g), \\ F^*(g, u) &:= \overline{F(g^{-1}, g^{-1}ug)}, \end{aligned}$$

or, entirely in terms of δ -functions,

$$\begin{aligned} (\delta_{g_1} \otimes f_1) \bullet (\delta_{g_2} \otimes f_2) &= \delta_{g_1 g_2} \otimes f_1(g_2(\cdot)g_2^{-1})f_2 \\ \Delta(\delta_g \otimes f)(g_1, u_1; g_2, u_2) &= \delta_g(g_1)\delta_g(g_2)f(u_1 u_2) \\ \epsilon(\delta_g \otimes f) &= f(e) \\ S(\delta_g \otimes f) &= \delta_{g^{-1}} \otimes f(g^{-1}(\cdot)^{-1}g), \\ (\delta_g \otimes f)^* &= \delta_{g^{-1}} \otimes f^*(g^{-1}(\cdot)g). \end{aligned}$$

In the following, we will use both the algebraic and the group convolution formulations. In the latter form it is less easy to take the limit to E_3 but see [5].

The momentum space is now the curved space $S^3 = SU_2$ with ‘translation Hopf algebra’ given by functions $C(SU_2)$. It acts on another copy of $C(SU_2)$, functions on momentum space, by pointwise multiplication. In a suitable formulation, the irreducible representations of $D(SU_2)$ are labelled by the SU_2 -conjugacy classes $C_m = \{v e^{im\lambda t_3} v^{-1} \mid v \in SU_2\}$ in the momentum space SU_2 and irreducible unitary representations Π_s of associated stabilizers $N_m = \{g \in SU_2 \mid g e^{im\lambda t_3} g^{-1} = e^{im\lambda t_3}\}$ [39]. Note that $C_0 = \{1\}$ and $C_{2\pi/\lambda} = \{-1\}$ and that all the other conjugacy classes are isomorphic to 2-spheres in the Lie algebra coordinate system, namely $|\vec{p}| = m$. Clearly $N_0 \simeq N_{2\pi/\lambda} \simeq SU_2$ and $N_m \simeq U(1)$ for generic values of m . In the generic case the carrier spaces for the irreducible representations are

$$V_{ms} = \{\psi : SU_2 \rightarrow \mathbb{C} \mid \psi(v e^{i\alpha t_3}) e^{s\alpha} \psi(v), \forall \alpha \in [0, 4\pi), v \in SU_2\}. \tag{49}$$

These are the same spaces of monopole sections as before for E_3 . An element $F \in D(SU_2)$ acts via

$$\Pi_{ms}(F)\psi(v) = \int dg F(g, g^{-1}v e^{im\lambda t_3} v^{-1}g)\psi(g^{-1}v).$$

The singular elements have the simple action

$$\Pi_{ms}(\delta_g \otimes f)\psi(v) = f(g^{-1}v e^{im\lambda t_3} v^{-1}g)\psi(g^{-1}v).$$

As for E_3 we can alternatively use carrier spaces which are spaces of vector-valued functions satisfying a constraint. Again we switch from the function $\psi \in V_{ms}$ to the vector-valued function defined as in (40) by

$$\tilde{\phi}(u) = \rho^s(v)\psi(v)|s, -s\rangle, \tag{50}$$

where now $u = v e^{im\lambda t_3} v^{-1} \in C_m$. They are spaces of sections of a monopole bundle over C_m with projection

$$e(u) = \rho^s(v)|s, -s\rangle\langle s, -s| \rho^s(v^{-1})$$

as before but now with the 2-sphere viewed as a conjugacy class $C_m \subset SU_2$ rather than as an orbit in su_2 . The functions (50) satisfy the group-valued analogue of the constraint (42),

$$\rho^s(u)\tilde{\phi}(u) = e^{-im\lambda s}\tilde{\phi}(u), \tag{51}$$

as one can check by an analogous calculation to the one carried out after (42).

For a unified description we now foliate $S^3 = SU_2$ as

$$\bigcup_{m \in (0, 2\pi/\lambda)} S_m^2 \simeq SU_2 \setminus \{1, -1\}.$$

Geometrically, $SU_2 \setminus \{1, -1\}$ is the 3-sphere without north and south poles, which we denote S_{NS}^3 . We define the space

$$W_s^1 = \{\tilde{\phi} : S_{NS}^3 \rightarrow \mathbb{C}^{2s+1}\}, \tag{52}$$

and impose a group-valued constraint (51). Then we obtain representations of $D(SU_2)$ on the spaces

$$W_{ms} = \{\tilde{\phi} : S_{NS}^3 \rightarrow \mathbb{C}^{2s+1} \mid \rho^s(u)\tilde{\phi}(u) = e^{-im\lambda s}\tilde{\phi}(u)\}, \tag{53}$$

essentially as before, while to obtain an irrep we may still have to impose a constraint that $\tilde{\phi}$ has support on C_m (we will give this in a different coordinate system shortly). For spins 1/2 and 1 this is automatic. The action of $D(SU_2)$ is most easily expressed in terms of the singular elements:

$$\Pi_{ms}(\delta_g \otimes f)\tilde{\phi}(u) = f(g^{-1}ug)\rho(g)\tilde{\phi}(g^{-1}ug).$$

In the case of the euclidean group we were able to apply a Fourier transform to obtain irreducible representations in terms of functions obeying a differential equation. We can do just the same in the non-Abelian case provided we use the modern tools of quantum group Fourier transform [3, 18, 19, 35]. If $\tilde{\phi}$ is a function on SU_2 we Fourier transform it to one on the noncommutative space $U(su_2)$ of the spin-model spacetime by

$$\phi(x) = \int_{SU_2} d^3 p J(\vec{p}) \tilde{\phi}(\vec{p}) \psi_{\vec{p}}(x),$$

using the noncommutative plane waves

$$\psi_{\vec{p}}(x) = e^{i\vec{p}\cdot\vec{x}}$$

in [3]. Here x_1, x_2 and x_3 are the generators of $U(su_2)$ with the commutation relations (27) discussed in section 3.2, and $d^3 p J(\vec{p})$ is the Haar measure on SU_2 in the Lie algebra coordinate system. The orbit spheres in these notations are

$$C_m = \{e^{i\lambda\vec{p}\cdot\vec{t}} \mid |\vec{p}| = m\}$$

so $m = |\vec{p}|$ defines the sphere, or equivalently

$$\mathcal{P}_0 = \cos(m\lambda/2)$$

in our global coordinates $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ of section 3.2 and in a patch where $\mathcal{P}_0 \geq 0$. Converting to the corresponding u provides the additional restriction on the spaces W_{ms} mentioned above as

$$\left(\frac{1}{2} \text{Tr}(u) - \cos(m\lambda/2)\right)\tilde{\phi} = 0.$$

Next, for spin 0 the constraint (51) on the field $\tilde{\phi}$ is empty as before and we have to separately impose the C_m relation as discussed,

$$\mathcal{P}_0\tilde{\phi} = \cos(m\lambda/2)\tilde{\phi}.$$

Under Fourier transform, multiplication by \mathcal{P}_0 becomes $1 - \iota \frac{\lambda}{2} \partial_0 = \sqrt{1 + \frac{\lambda^2}{4} \Delta}$ in terms of the noncommutative partial derivatives on the noncommutative spacetime. These were introduced in [3] but see also [19, 35] (but note the use of λ there in the role of $\lambda/2$ in our conventions). All we need to know about the noncommutative differentials ∂_a for the present purposes is that they diagonalize the noncommutative plane waves $\psi_{\vec{p}}(x)$ with eigenvalues $\iota \mathcal{P}_a$. Here $\Delta = \partial_a \partial^a$ is the noncommutative Laplace operator. So the noncommutative scalar wave equation is

$$\left(\Delta + \left(\frac{\sin(m\lambda/2)}{\lambda/2} \right)^2 \right) \phi = 0.$$

This agrees with [3] for a suitable interpretation of the effective mass.

For spin 1/2 the constraint (51) is

$$e^{\frac{\lambda}{2} \vec{p} \cdot \vec{\sigma}} \tilde{\phi} = e^{-\frac{\lambda}{2} m \lambda} \tilde{\phi}.$$

Using our global coordinates, this comes out as

$$\left(\mathcal{P}_0 + \iota \frac{\lambda}{2} \vec{P} \cdot \vec{\sigma} \right) \tilde{\phi} = e^{-\frac{\lambda}{2} m \lambda} \tilde{\phi}$$

Squaring, using the identity $\mathcal{P}_0^2 + \frac{\lambda^2}{4} \vec{P}^2 = 1$ and the constraint equation again to replace $\iota \vec{P} \cdot \vec{\sigma}$, gives the C_m relations (so these do not need to be imposed separately). Next, using these relations we have

$$\cos(m\lambda/2) \tilde{\phi} + \iota \frac{\lambda}{2} \vec{P} \cdot \vec{\sigma} \tilde{\phi} = (\cos(m\lambda/2) - \iota \sin(m\lambda/2)) \tilde{\phi}$$

and cancel to obtain

$$\left(\vec{P} \cdot \vec{\sigma} + \frac{\sin(m\lambda/2)}{\lambda/2} \right) \tilde{\phi} = 0$$

as the noncommutative Dirac equation in momentum space. This equation squares to give $\vec{P}^2 = \frac{\sin^2(m\lambda/2)}{\lambda^2/4}$ which is equivalent to the C_m relation so this is all we need to impose to obtain the irreducible representation. The equation after Fourier transform becomes

$$\left(\iota \vec{\partial} \cdot \vec{\sigma} - \frac{\sin(m\lambda/2)}{\lambda/2} \right) \phi = 0$$

as the noncommutative Dirac operator for the spin model. This agrees with [3] for our interpretation of the effective mass.

For spin 1, we use the adjoint representation of SU_2 . The constraint equation (51) is linear in $\tilde{\phi}$ so we can use any basis we choose and here we choose the Cartesian one and accordingly work with $\tilde{\phi} \cdot \vec{\sigma}$. Then the constraint equation becomes

$$\left(\mathcal{P}_0 + \iota \frac{\lambda}{2} \vec{P} \cdot \vec{\sigma} \right) \tilde{\phi} \cdot \vec{\sigma} \left(\mathcal{P}_0 - \iota \frac{\lambda}{2} \vec{P} \cdot \vec{\sigma} \right) = e^{-\iota m \lambda} \tilde{\phi} \cdot \vec{\sigma}$$

or

$$\left(\mathcal{P}_0^2 \tilde{\phi} \cdot \vec{\sigma} + \iota \frac{\lambda}{2} \mathcal{P}_0 [\vec{P} \cdot \vec{\sigma}, \tilde{\phi} \cdot \vec{\sigma}] + \frac{\lambda^2}{4} \vec{P} \cdot \vec{\sigma} (\tilde{\phi} \cdot \vec{P} + \iota \tilde{\phi} \times \vec{P} \cdot \vec{\sigma}) \right) = e^{-\iota m \lambda} \tilde{\phi} \cdot \vec{\sigma},$$

which comes out as

$$\left(1 - \frac{\lambda^2}{2} \vec{P}^2 \right) \tilde{\phi} - \lambda \mathcal{P}_0 \vec{P} \times \tilde{\phi} + \frac{\lambda^2}{2} (\vec{P} \cdot \tilde{\phi}) \vec{P} = e^{-\iota m \lambda} \tilde{\phi}.$$

We apply $\vec{P} \cdot ()$ to both sides and conclude that

$$\vec{P} \cdot \tilde{\phi} = 0.$$

In spacetime this becomes $\vec{\partial} \cdot \phi = 0$ in terms of the noncommutative partial derivatives. The constraint equation meanwhile reduces to

$$(2\mathcal{P}_0^2 - 1 - e^{-im\lambda})\tilde{\phi} = \lambda\mathcal{P}_0\vec{\mathcal{P}} \times \tilde{\phi}$$

on replacement of $1 - \frac{\lambda^2}{2}\vec{\mathcal{P}}^2$. Applying $\vec{\mathcal{P}} \times$ to this gives

$$(2\mathcal{P}_0^2 - 1 - e^{-im\lambda})\vec{\mathcal{P}} \times \tilde{\phi} = \frac{4}{\lambda}\mathcal{P}_0(\mathcal{P}_0^2 - 1)\tilde{\phi}$$

on the same replacement. Eliminating $\vec{\mathcal{P}} \times \tilde{\phi}$ between these equations gives an equation for \mathcal{P}_0 on $\tilde{\phi}$ which turns out to be our C_m relation in the wave operator form. Finally, going back to what remained of our constraint equation and replacing $\mathcal{P}_0^2 = \cos^2(m\lambda/2)$ gives

$$\vec{\mathcal{P}} \times \tilde{\phi} - i\frac{\sin(m\lambda/2)}{\lambda/2}\tilde{\phi} = 0$$

which together with our divergence condition provides the full content of the constraint equation (one may square it to get the C_m relation once again). Applying the Fourier transform gives

$$\vec{\partial} \times \phi + \frac{\sin(m\lambda/2)}{\lambda/2}\phi = 0$$

as our spin 1 wave equation, in agreement with [3] in the massless case discussed there.

Note that in all these equations, in momentum space the equations in terms of the Lie coordinates \vec{p} become the same as in the E_3 case, since the Lie and global coordinates are related by rescaling with $\frac{\sin(m\lambda/2)}{m\lambda/2}$, where $m = |\vec{p}|$. However, in the noncommutative geometry of $U(su_2)$ it is \mathcal{P}_a that appear as the natural partial derivatives, see [3, 35].

4.3. Representations of $SU_2 \times SU_2$

In this section, we show that the space (52) with a differential instead of a multiplicative constraint also carries all the irreducible representations of $SU_2 \times SU_2$. This is the semidual model to the $D(SU_2)$ model of the preceding section but we shall see that the irreps have a parallel construction. We denote the generators of the two copies of su_2 by J_a^L and J_a^R ; the Lie brackets are, in our conventions (A.1),

$$[J_a^L, J_b^L] = i\epsilon_{abc}J_c^L, \quad [J_a^R, J_b^R] = i\epsilon_{abc}J_c^R, \quad [J_a^L, J_b^R] = 0. \tag{54}$$

The irreps of this Lie algebra are well known to be labelled by two non-negative half-integer spins, which we call k and l , and to have dimension $(2k + 1)(2l + 1)$. There are two Casimirs

$$(J^R)^2 = \sum_{a=1}^3 (J_a^R)^2 \quad \text{and} \quad (J^L)^2 = \sum_{a=1}^3 (J_a^L)^2,$$

which take the following values on the irreps

$$(J^R)^2 = k(k + 1), \quad (J^L)^2 = l(l + 1), \quad k, l \in \frac{1}{2}(\mathbb{N} \cup 0). \tag{55}$$

We first show that one may realize these operators and their eigenvalues on the space

$$W_s = \{\tilde{\phi} : S^3 \rightarrow \mathbb{C}^{2s+1}\}$$

of all \mathbb{C}^{2s+1} -valued functions on S^3 . As before, we let ρ^s be the spin s representation so that $(\rho^s(t))^2 := \sum_{a=1}^3 \rho^s(t_a)\rho^s(t_a)$ has eigenvalue $s(s + 1)$. We define actions of the generators on W_s as

$$J_a^L = i\xi_a^L + \rho^s(t_a), \quad J_a^R = i\xi_a^R, \tag{56}$$

where ξ_a^L and ξ_a^R are the left- and right-generated vector fields associated with the generators t_a of su_2 as defined in (A.2). Squaring, we note that

$$(J^L)^2 = (J^R)^2 + 2t\xi_a^L\rho^s(t_a) + s(s+1)$$

so that (55) becomes

$$(J^R)^2\phi = k(k+1)\phi, \quad \phi \in W_s \tag{57}$$

and, with $l = s + k$,

$$(t\xi_a^L\rho^s(t_a) - ks)\phi = 0, \quad \phi \in W_s. \tag{58}$$

This is our ‘wave equation’ in mathematical terms, i.e. we obtain a representation on

$$W_{ks} = \{ \phi : S^3 \rightarrow \mathbb{C}^{2s+1} \mid (t\xi_a^L\rho^s(t_a) - ks)\phi = 0 \}$$

by imposing this constraint. We still need to impose the condition (57) separately in order to obtain an irrep, although this is automatic for spin 1/2 and 1 as we shall see shortly. The reason that we then obtain irreps is as follows. We start with the Peter–Weyl decomposition of $C(SU_2)$ (or rather L^2 in a Hilbert space context) in terms of matrix elements of irreps V_k of SU_2 . This decomposes the function space into irreducible blocks $V_k \otimes V_k^*$ where J^L, J^R act on the left and right factors, respectively. This is the decomposition provided by the ‘wave equation’

$$((\xi^R)^2 + k(k+1))\phi = 0 \tag{59}$$

on scalar fields (the Laplace–Beltrami equation on S^3). Now in our case we have \mathbb{C}^{2s+1} -valued fields,

$$W_s = \mathbb{C}^{2s+1} \otimes (\oplus_k (V_k \otimes V_k^*)) = \oplus_k (\mathbb{C}^{2s+1} \otimes V_k) \otimes V_k^*$$

where J^R acts on V_k^* as before and J^L acts on $\mathbb{C}^{2s+1} \otimes V_k$. The former is an irrep of SU_2 but the latter is not. The constraint (58) picks out an irrep of total spin $l = s + k$ within it. Hence it picks out a block $V_l \otimes V_k^*$ within W_s as isomorphic to our constrained function space W_{ks} if we also impose (59). Hence these are indeed irreps and of the expected size.

It is again interesting to investigate the constraint (58) for low values of s . For $s = \frac{1}{2}$ we obtain

$$t\sigma_a\xi_a^L\phi = k\phi. \tag{60}$$

Applying $-t\sigma_a\xi_a^L$ to both sides gives

$$(-(J^R)^2 + i\sigma_a\xi_a^L)\phi = -ik\sigma_a\xi_a^L\phi$$

or (59).

For $s = 1$ we again use the Cartesian representation (34) to obtain

$$\epsilon_{abc}\xi_a^L\phi_c = k\phi_b. \tag{61}$$

Acting with ξ_b^L and summing over c gives

$$-\frac{1}{2}\epsilon_{abc}[\xi_a^L, \xi_b^L]\phi_c = k\xi_c^L\phi_c \quad \Leftrightarrow \quad -\xi_c^L\phi_c = k\xi_c^L\phi_c.$$

Since $k > 0$ we conclude

$$\xi_c^L\phi_c = 0.$$

Applying $\epsilon_{deb}\xi_d^L$ to both sides of (61) now gives

$$(J^R)^2\phi_e + \xi_d^L\xi_e^L\phi_d = k^2\phi_e.$$

Now use

$$\xi_d^L\xi_e^L\phi_d = \xi_e^L\xi_d^L\phi_d + [\xi_d^L, \xi_e^L]\phi_d = -k\phi_e$$

to conclude (59) again. Thus, like in the euclidean case, only the linear constraint (58) needs to be imposed for $s = \frac{1}{2}$ and $s = 1$.

This concludes our wave-equation picture of the representation theory at a mathematical level. In terms of physical variables we can understand the above as follows. We again use a parameter λ in parametrizing the SU_2 where the fields live, but note that this is now position space and that the value of the parameter in our physical picture is now $\lambda = 1/l_c$. This is the semidual of the model in the preceding section but like that one, it is a (different) deformation of the self-dual E_3 model, recovered as $\lambda \rightarrow 0$.

Let us note first of all that the actual semidual, as explained in section 3.3, is $SU_2 \ltimes SU_2$ by the right adjoint action, which is isomorphic to the above group $SU_2 \times SU_2$. Denoting the generators of the former by J_a , P_a for the two copies respectively, their commutation relations were given in (28) and their relations to the generators (54) are

$$P_a = \lambda J_a^R, \quad J_a = J_a^R + J_a^L \quad \text{or} \quad J_a^L = J_a - \frac{P_a}{\lambda}, \quad J_a^R = \frac{P_a}{\lambda}.$$

The physical Casimirs are

$$P^2 = \lambda^2 (J^R)^2, \quad C = \vec{p} \cdot \vec{j} - \frac{\lambda}{2} J^2 = \frac{\lambda}{2} ((J^R)^2 - (J^L)^2).$$

As before, we use the same relations with p_a in place of P_a when we refer to the (noncommutative) momentum space with these as coordinates.

With the definitions (56), the action of the angular momentum J_a on the space W_s is

$$J_a = \iota \text{Ad}_a + \rho^s(t_a),$$

where $\text{Ad}_a = \xi_a^L + \xi_a^R$ is the adjoint action as a vector field on the group in terms of vector fields for the left and right action (A.2) on $SU_2 = S^3$. This becomes the usual orbital angular momentum on \mathbb{R}^3 in the limit $\lambda \rightarrow 0$. The action of P_a is

$$P_a = \iota \lambda \xi_a^R$$

and the associated Casimir is the Laplace–Beltrami operator on S^3 . Its eigenvalues (the squared mass of the particle) are, according to (59), given by

$$P^2 \phi = \lambda^2 k(k+1) \phi, \tag{62}$$

so essentially $m = \lambda k$ is the mass of the particle.

Next, a short computation gives

$$C = -\iota \lambda \xi_a^L \rho^s(t)_a - \frac{\lambda}{2} s(s+1).$$

In line with what we have done before, we therefore impose a suitable value of this as a further ‘wave operator’ to obtain representations of $SU(2) \times SU(2)$ on the spaces

$$W_{ks} = \left\{ \phi : S^3 \rightarrow \mathbb{C}^{2s+1} \mid \left(\vec{p} \cdot \vec{j} - \frac{\lambda}{2} J^2 + \lambda ks + \frac{\lambda}{2} s(s+1) \right) \phi = 0 \right\},$$

which are irreps at least for spin 1/2 and spin 1. For higher spin we need to impose (59) as well. Taking the limit $\lambda \rightarrow 0$ while keeping the mass $m = \lambda k$ fixed reproduces the constraint (45) in euclidean space, as required.

Note that these computations are done in position space. In terms of our previous exposition, we have gone from noncommutative momentum space (functions of p_a) to position space (functions on SU_2) again by means of the quantum group Fourier transform, this time read the other way. The only fact we need to know is that left multiplication by p_a becomes the vector field $-\iota \xi_a^L$ while right-multiplication by p_a becomes the vector field $\iota \xi_a^R$. If one

wants to do things in the noncommutative momentum space then the constraint (58) appears as

$$(\rho^s(t_a)p_a + ms)\tilde{\phi}(\vec{p}) = 0.$$

We distinguish here between the generators P_a of the isometry group and the noncommutative coordinates p_a on momentum space. They are both copies of the scaled su_2 Lie algebra relations as stated for P_a above.

Note that our ‘orbits’ or conjugacy classes in momentum space still exist as before, but now as ‘fuzzy spheres’ of radius $m = \lambda k$ in this momentum space instead of usual spheres as for the E_3 model. It is known how to construct monopole sections in this context (as projective modules) but we are not aware of a full analogue of the Hopf fibration itself, hence the ‘upstairs’ point of view with field ψ as in (49) requires further elaboration using methods of noncommutative geometry. The downstairs picture of the monopole sections is defined for $s = 1/2$ by projections

$$e(\vec{p}) = \frac{k+1}{2(k+\frac{1}{2})} + \frac{t_a P_a}{\lambda(k+\frac{1}{2})}.$$

One can check that $e^2 = e$ using the P_a commutation relations (28) and the constraint (62). As $\lambda \rightarrow 0$ and $k \rightarrow \infty$ with $m = \lambda k$ fixed we see that we recover the standard monopole projector given in (44).

5. Discussion

We have seen that the ‘particle content’ in the E_3 flat spacetime model can be deformed in two ways, one with the mass m ‘compressed’ by the sine function as momentum space is compactified to SU_2 but otherwise similar (the spin model) and the other with mass m discretized in units of λ due to a fuzzy sphere in momentum space (the $SU_2 \times SU_2$ model). Thus although the physical parameters for the irreps in the two models are very different the actual constructions of the irreps are similar and in some sense the physical states ‘correspond’ through their common limit (i.e. with arbitrary accuracy as the relevant $\lambda \rightarrow 0$) even though they are different. This is the ‘remnant’ of the self-duality in the degenerate cases that we have looked at (the upper part of figure 1).

This picture also applies elsewhere in figure 1 and can, in principle, be developed entirely analogously. Thus the $SL_2(\mathbb{C})$ model of section 3.4 is similar in principle to the $SU_2 \times SU_2$ model of section 4.3 while its semidualization is the bicrossproduct model. Its representation theory, as a semidirect product algebra, is readily developed in the same manner as for the quantum double in section 4.2. The difference is that the adjoint action is replaced by a nonlinear action deforming it as we have explained in section 3.5. In both cases we have complications due to the non-compactness. The ‘quantum gravity with cosmological constant’ case of section 3.6 can similarly be developed—with a lot more effort—as a q -deformation of section 4.2. Here again we see that the irreps on one hand are those of quantum $SL_2(\mathbb{C})$ and on the other hand in the semidual model, they are irreps of quantum $SU_2 \times SU_2$ —described by the same parameters as in the non- q -deformed case and with the same features of continuous and discrete parameters being ‘matched’ in a limiting sense. How this proceeds given that the signatures (expressed in the $*$ -structures) are very different remains to be seen. Roughly speaking, we expect that the algebraic equivalence of categories ignoring the $*$ -structures explained in section 3.6 is complemented by two different ‘cross sections’ consisting of the unitary irreps in each model, and that these slices are in some sense ‘transverse’.

We can gain some insight again from the simplest E_3 case. Thus here on one hand we have irreps of E_3 constructed as monopole sections over spheres and a dual model in which the

irreps are constructed by wave equations in \mathbb{R}^3 . In a fixed point of view these are respectively momentum and position space treatments but from the point of view in which each theory is considered the primary one, they are both (say) position space representations. Thus we consider functions $\phi(\vec{x})$ with values in \mathbb{C}^{2s+1} and consider both our possible constraints as two different physical models on this position space \mathbb{R}^3 . One is related to the operator $\rho^s(t_a)\partial_a$ and the other to the operator $\rho^s(t_a)x_a$. It is interesting to note that for $s = 1/2$ these two are closely related to the Riemannian geometry of the sphere. Thus,

$$[\vec{t} \cdot \vec{x}, \vec{t} \cdot \nabla] = it \cdot (\vec{x} \times \nabla) - \frac{3}{4},$$

using elementary properties of the Pauli matrices. Now the expression on the right is essentially a massive Dirac operator on a sphere S^2 with its standard Riemannian metric. (It commutes with x^2 and hence defines an operator on \mathbb{C}^2 -valued functions on the sphere.) Thus Riemannian geometry arises here out of the interaction of the system and the dual system. Also, we see that our two operators form some kind of ‘Heisenberg pair’ with the curved Dirac operator in the role of Planck’s constant. In this sense, our two methods of extracting irreps of E_3 are ‘transverse’ and describe different physics if one views both in position space, in the sense that one cannot simultaneously restrict to both: restricting to an irrep in one point of view should typically have inner products with all the irreps in the other point of view. We expect that this is part of the story for the full quantum gravity case.

Appendix. Vector fields and forms on Lie groups

Here we collect some facts about forms and vector fields on an n -dimensional Lie group G , which are used in the main text. In order to simplify notation we assume G to be a matrix group. We write \mathfrak{g} for the Lie algebra of G and work with generators for which the structure constants are purely imaginary. With the notation $t_a, a = 1, \dots, n$, for the generators the Lie brackets take the form

$$[t_a, t_b] = \iota f_{ab}^c t_c, \tag{A.1}$$

where f_{ab}^c are real and we use the convention that repeated indices are summed over. It follows that the structure constants are f_{ab}^c in terms of the ‘real’ generators $-it_a$; the reader may find it useful to read some of the geometrical formulae in this paper in terms of these generators. Associated with the generators t_a we have the left-generated vector fields ξ_a^L and the right-generated vector fields ξ_a^R , defined via

$$\xi_a^L f(g) = \left. \frac{d}{ds} \right|_{s=0} f(e^{st_a} g), \quad \xi_a^R f(g) = \left. \frac{d}{ds} \right|_{s=0} f(g e^{-st_a}). \tag{A.2}$$

They close under the Lie bracket of vector fields and give two commuting copies of \mathfrak{g} :

$$[\xi_a^L, \xi_b^L] = f_{ab}^c \xi_c^L, \quad [\xi_a^R, \xi_b^R] = f_{ab}^c \xi_c^R, \quad [\xi_a^L, \xi_b^R] = 0$$

Using the matrix structure of G we can identify T_{gG} with matrices of the form $g\xi$, where $\xi \in \mathfrak{g}$, or with matrices of the form ξg . Then we can also write

$$\xi_a^L(g) = \iota t_a g, \quad \xi_a^R(g) = -\iota g t_a. \tag{A.3}$$

Using either of the definitions (A.2) and (A.3) it is easy to see that the left-generated vector fields are invariant under the right action $R_h : g \mapsto gh$ of G on itself (and hence on TG) and that the right-generated vector fields are invariant under the left-action $L_h : g \mapsto hg$ of G on itself. We have the following relation between left- and right-generated vector fields:

$$(L_g R_{g^{-1}})'(\xi_a^L(g)) = -\xi_a^R(g).$$

With the abbreviation

$$\text{Ad}(g)(t_a) = gt_ag^{-1} = R^b_a(g)t_b$$

it follows that

$$\xi_a^R(g) = -R^b_a(g)\xi_b^L(g).$$

There is as basis of 1-forms dual to the above vector fields which can be obtained by expanding the Maurer–Cartan form

$$\theta = g^{-1} dg.$$

The Maurer–Cartan form is Lie-algebra valued and manifestly left-invariant. Expanding in the Lie algebra basis t_a , $a = 1, \dots, n$, we obtain a basis $\sigma^{R,a}$ of left-invariant 1-forms

$$g^{-1} dg = -\iota_{t_a} \sigma^{R,a}. \tag{A.4}$$

The 1-forms $\sigma^{R,a}$ are dual to the left-invariant (and right-generated) vector fields ξ_a^R :

$$\sigma^{R,a}(\xi_b^R) = \delta^a_b.$$

We obtain right-invariant 1-forms σ_a^L by expanding

$$-g d(g^{-1}) = dg g^{-1} = \iota_{t_a} \sigma^{L,a} \tag{A.5}$$

with the duality relation

$$\sigma^{L,a}(\xi_b^L) = \delta^a_b.$$

Comparing (A.4) with (A.5) we have the relation

$$\sigma^{L,a} = -R^a_b \sigma^{R,b}.$$

Since the Maurer–Cartan form satisfies

$$d\theta + \theta \wedge \theta = 0$$

we deduce

$$d\sigma^{R,a} = -\frac{1}{2} f^a_{bc} \sigma^{R,b} \wedge \sigma^{R,c}$$

and by a similar argument

$$d\sigma^{L,a} = -\frac{1}{2} f^a_{bc} \sigma^{L,b} \wedge \sigma^{L,c}.$$

We note that every compact Lie group has a bi-invariant Riemannian metric. In terms of the 1-forms introduced above it can be written as

$$ds^2 = \kappa_{ab} \sigma^{R,a} \sigma^{R,b} = \kappa_{ab} \sigma^{L,a} \sigma^{L,b}, \tag{A.6}$$

where κ is the Killing form on the Lie algebra i.e.

$$\kappa_{ab} = -\text{tr}(\text{ad}(t_a)\text{ad}(t_b)).$$

The Laplace operator associated with this metric can be written in terms of the inverse metric κ^{ab} and either the left- or right-generated vector fields as

$$\kappa^{ab} \xi_a^R \xi_b^R = \kappa^{ab} \xi_a^L \xi_b^L. \tag{A.7}$$

Finally, although the tangent bundle of any Lie group is isomorphic to the trivial bundle $G \times \mathfrak{g}$, this is not canonical in the sense that we can use either the left- or the right-translations to trivialize the bundle. In the left-trivialization, $g\xi \in T_g G$ is identified with $\xi \in \mathfrak{g}$. In the right-trivialization $\xi g \in T_g G$ is identified with $\xi \in \mathfrak{g}$. Both left- and right-translation can also be used to define a connection on TG . Both the connections are flat. In the left-trivialization, the connection defined by the left-translation has the covariant derivative

$D_L = d$. The right-translation has the covariant derivative $D_R = d + g^{-1} dg$. Note that $D_R^2 = 0$, as required for flatness. The Levi-Civita connection (unique torsion free connection which preserves the Killing metric (A.6) turns out to be the average of the connection for the left- and right-translation. In the left-trivialization the Levi-Civita connection 1-form is therefore $A_{LC} + \frac{1}{2}g^{-1} dg$, leading to the covariant derivative $D_{LC} = d + \frac{1}{2}g^{-1} dg$. The Levi-Civita connection is not flat. Its curvature is

$$F_{LC} = d\left(\frac{1}{2}g^{-1} dg\right) + \frac{1}{4}g^{-1} dg \wedge g^{-1} dg = -\frac{1}{4}g^{-1} dg \wedge g^{-1} dg.$$

References

- [1] 't Hooft G 1996 Quantization of point particles in (2+1)-dimensional gravity and space-time discreteness *Class. Quantum Grav.* **13** 1023–39
- [2] Matschull H J and Welling M 1998 Quantum mechanics of a point particle in (2+1)-dimensional gravity *Class. Quantum Grav.* **15** 2981–3030
- [3] Batista E and Majid S 2003 Noncommutative geometry of angular momentum space $U(su_2)$ *J. Math. Phys.* **44** 107–37
- [4] Bais F A and Muller N M 1998 Topological field theory and the quantum double of $SU(2)$ *Nucl. Phys. B* **530** 349–400
- [5] Schroers B J 2001 Combinatorial quantization of Euclidean gravity in three dimensions *Quantization of Singular Symplectic Quotients* ed N P Landsman, M Pflaum and M Schlichenmaier pp 307–28 (Basel: Birkhäuser Verlag)
- [6] Freidel L and Livine E R 2006 Ponzano–Regge model revisited: III. Feynman diagrams and effective field theory *Class. Quantum Grav.* **23** 2021–62
- [7] Majid S 1991 Hopf–von Neumann algebra bicrossproducts, Kac algebra bicrossproducts, and the classical Yang–Baxter equations *J. Funct. Anal.* **95** 291–319
- [8] Majid S and Ruegg H 1994 Bicrossproduct structure of the κ -Poincaré group and noncommutative geometry *Phys. Lett. B* **334** 348–54
- [9] Lukierski J, Nowicki A, Ruegg H and Tolstoy V N 1991 q -Deformation of Poincaré algebra *Phys. Lett. B* **268** 331–8
- [10] Meusburger C and Schroers B J 2009 Generalised Chern–Simons actions for 3D gravity and κ -Poincaré symmetry *Nucl. Phys. B* **806** 462–88
- [11] Majid S 1991 Examples of braided groups and braided matrices *J. Math. Phys.* **32** 3246–53
- [12] Majid S 1995 *Foundations of Quantum Group Theory* (Cambridge: Cambridge University Press)
- [13] Freidel L, Noui K and Roche P 2007 $6J$ symbols duality relations *J. Math. Phys.* **48** 113512
- [14] Schroers B J 2007 Lessons from (2+1)-dimensional quantum gravity *Proc. PoS (QG-Ph) 035 for Workshop From Quantum to Emergent Gravity: Theory and Phenomenology (Trieste)*
- [15] Buffenoi E, Noui K and Roche P 2002 Hamiltonian quantization of Chern–Simons theory with $SL(2, \mathbb{C})$ group *Class. Quantum Grav.* **19** 4953–5015
- [16] Majid S 1990 Physics for algebraists: noncommutative and noncocommutative Hopf algebras by a bicrossproduct construction *J. Algebr.* **130** 17–64
- [17] Majid S 1991 The principle of representation-theoretic self-duality *Phys. Essays* **4** 395–405
- [18] Majid S 1995 Duality principle and braided geometry *Springer Lect. Notes Phys.* **447** 125–44
- [19] Freidel L and Majid S 2008 Noncommutative harmonic analysis, sampling theory and the Duflo map in 2+1 quantum gravity *Class. Quantum Grav.* **25** 045006
- [20] Majid S 1990 Matched pairs of Lie groups associated to solutions of the Yang–Baxter equations *Pac. J. Math.* **141** 311–32
- [21] Semenov-Tian-Shansky M A 1994 Poisson Lie groups, quantum duality principle, and the quantum double *Contemporary Mathematics. Mathematical Aspects of Conformal and Topological Field Theories and Quantum Groups* vol 175 ed P J Sally Jr et al (Providence, RI: American Mathematical Society) pp 219–84
- [22] Achúcarro A and Townsend P 1986 A Chern–Simons action for three-dimensional anti-de Sitter supergravity theories *Phys. Lett. B* **180** 89
- [23] Witten E 1988 2+1 Gravity as an exactly soluble system *Nucl. Phys. B* **311** 46–78
- [24] Fock V V and Rosly A A 1992 Poisson structures on moduli of flat connections on Riemann surfaces and r -matrices *ITEP 72-92* (arXiv:math.QA/9802054)
- [25] Alekseev A Y, Grosse H and Schomerus V 1995 Combinatorial quantization of the Hamiltonian Chern–Simons theory *Commun. Math. Phys.* **172** 317–58

- [26] Alekseev A Y, Grosse H and Schomerus V 1995 Combinatorial quantization of the Hamiltonian Chern–Simons theory II *Commun. Math. Phys.* **174** 561–604
- [27] Alekseev A Y and Schomerus V 1996 Representation theory of Chern–Simons observables *Duke Math. J.* **85** 447–510
- [28] Alekseev A Y and Malkin A Z 1995 Symplectic structure of the moduli space of flat connections on a Riemann surface *Commun. Math. Phys.* **169** 99–119
- [29] Meusburger C and Schroers B J 2003 Poisson structure and symmetry in the Chern–Simons formulation of 2+1 dimensional gravity *Class. Quantum Grav.* **20** 2193–234
- [30] Martin S 1989 Observables in 2+1 dimensional gravity *Nucl. Phys. B* **327** 178
- [31] Meusburger C and Schroers B J 2006 Phase space structure of Chern–Simons theory with a non-standard puncture *Nucl. Phys. B* **738** 425–56
- [32] Meusburger C and Schroers B J 2005 Boundary conditions and symplectic structure in the Chern–Simons formulation of 2+1 dimensional gravity *Class. Quantum Grav.* **22** 3689–724
- [33] Meusburger C and Schroers B J 2004 The quantization of Poisson structures arising in Chern–Simons theory with gauge group $G \ltimes \mathfrak{g}^*$ *Adv. Theor. Math. Phys.* **7** 1003–43
- [34] Drinfeld V G 1987 Quantum groups *Proc. ICMS* (Providence, RI: American Mathematical Society)
- [35] Majid S 2005 Noncommutative model with spontaneous time generation and Planckian bound *J. Math. Phys.* **46** 103520
- [36] Majid S 1994 q -Euclidean space and quantum Wick rotation by twisting *J. Math. Phys.* **35** 5025–34
- [37] Gomez X and Majid S 2002 Noncommutative cohomology and electromagnetism on $\mathbb{C}q[S_2]$ at roots of unity *Lett. Math. Phys.* **60** 221–37
- [38] Lyubashenko V and Majid S 1994 Braided groups and quantum Fourier transform *J. Algebr.* **166** 506–28
- [39] Koornwinder T and Muller N 1997 The quantum double of a (locally) compact group *J. Lie Theory* **7** 33–52